Numerical Integration (Quadrature)

Math 426

University of Alaska Fairbanks

November ~~[illegible]~~, 2020
11

Motivation

In calculus class we tricked you. We made you belive that you had the power to compute definite integrals. But your powers are fragile. It's easy to write down integrals where you can't find an antiderivative and therefore can't write down the exact value.

E.g.

$$\int_0^1 \sin(\sqrt{x})\, dx$$

Motivation

In calculus class we tricked you. We made you belive that you had the power to compute definite integrals. But your powers are fragile. It's easy to write down integrals where you can't find an antiderivative and therefore can't write down the exact value.

E.g.

$$\int_0^1 \sin(\sqrt{x})\, dx$$

$$\int_a^x \sin(\sqrt{s})\, ds$$

You need to find $F(x)$ with

$$F'(x) = \sin(\sqrt{x})$$

in which case

$$\int_0^1 \sin(\sqrt{x})\, dx = F(1) - F(0.)$$

Motivation

In calculus class we tricked you. We made you belive that you had the power to compute definite integrals. But your powers are fragile. It's easy to write down integrals where you can't find an antiderivative and therefore can't write down the exact value.

E.g.

$$\int_0^1 \sin(\sqrt{x})\, dx$$

You need to find $F(x)$ with

$$F'(x) = \sin(\sqrt{x})$$

in which case

$$\int_0^1 \sin(\sqrt{x})\, dx = F(1) - F(0.)$$

Good luck!

Strategy

We want to compute

$$\int_a^b f(x)\, dx$$

Strategy

We want to compute

$$\int_a^b f(x)\,dx$$

We find an approximating function $g(x)$ instead where we **can** compute definite integrals and compute

$$\int_a^b g(x)\,dx$$

Strategy

We want to compute

$$\int_a^b f(x)\,dx$$

We find an approximating function $g(x)$ instead where we **can** compute definite integrals and compute

$$\int_a^b g(x)\,dx$$

An important part of this process will be to estimate the error

$$E = \left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| = \left| \int_a^b (f(x) - g(x))\,dx \right|$$

$$b - a = h$$

Via Linear Interpolation (AKA Trapezoid Rule)

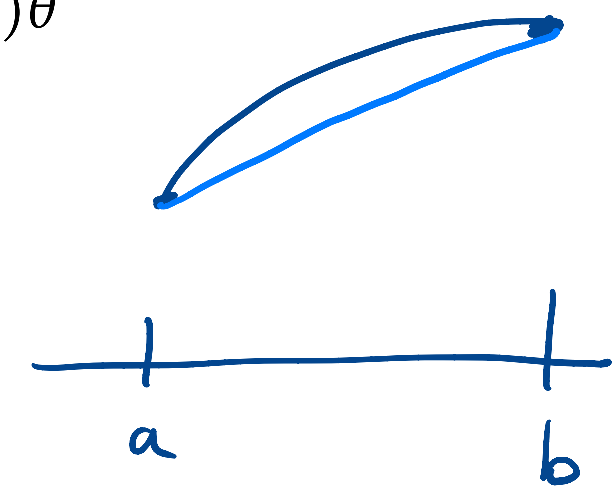
Let $g(x)$ be the linear interpolant

$$g(x) = f(a)(1-\theta) + f(b)\theta$$

with

$$\theta = \frac{x-a}{b-a} = \frac{x-a}{h}$$

$\theta(x)$



Via Linear Interpolation (AKA Trapezoid Rule)

Let $g(x)$ be the linear interpolant

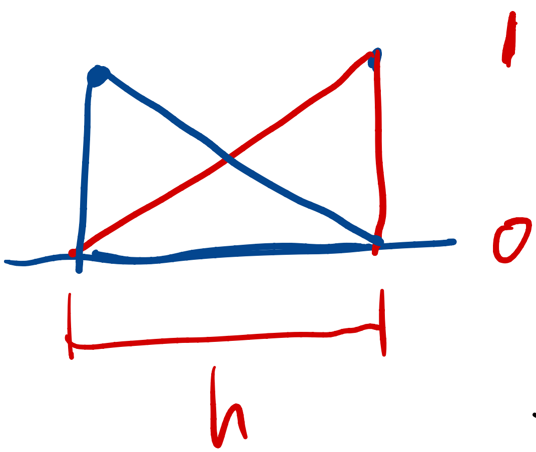
$$g(x) = f(a)(1-\theta) + f(b)\theta$$

with

$$\theta = \frac{x-a}{b-a} = \frac{x-a}{h}$$

$$d\theta = \frac{dx}{h}$$

$$dx = h\,d\theta$$



$\theta(x)$

$$\int_a^b \theta\, dx = h\int_0^1 \theta\, d\theta = h\left.\frac{\theta^2}{2}\right|_0^1 = \frac{h}{2}.$$

$$\int_a^b (1-\theta)\, dx = h\int_0^1 (1-\theta)\, d\theta = -h\left.\frac{(1-\theta)^2}{2}\right|_0^1 = \frac{h}{2}.$$

Via Linear Interpolation (AKA Trapezoid Rule)

Let $g(x)$ be the linear interpolant

$$g(x) = f(a)(1-\theta) + f(b)\theta$$

$\frac{a+b}{2}$

with

$$\theta = \frac{x-a}{b-a} = \frac{x-a}{h}$$

$$\int_a^b \theta \, dx = h \int_0^1 \theta \, d\theta = h \left. \frac{\theta^2}{2} \right|_0^1 = \frac{h}{2}.$$

$$\int_a^b (1-\theta) \, dx = h \int_0^1 (1-\theta) \, d\theta = -h \left. \frac{(1-\theta)^2}{2} \right|_0^1 = \frac{h}{2}.$$

$$\int_a^b g(x) \, dx = h \frac{f(a) + f(b)}{2}$$

Error for Trapezoidal Rule

$$f(x) = g(x) + \frac{f''(\xi)}{2}(x-a)(x-b)$$

Error for Trapezoidal Rule

$$f(x) = g(x) + \frac{f''(\xi)}{2}(x-a)(x-b)$$

Mean Value Theorem For integrals

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f''(\xi(x))(x-a)(x-b)\,dx$$

$$= f''(c) \int_a^b (x-a)(x-b)\,dx$$

$$\frac{1}{b-a}\int_a^b f(s)\,ds = f(c)$$

$$\int_a^b f(s)\,ds = f(c)\int_a^b 1\,ds$$

Error for Trapezoidal Rule

$$f(x) = g(x) + \frac{f''(\xi)}{2}(x-a)(x-b)$$

$$\int_a^b (f(x) - g(x))\, dx = \int_a^b f''(\xi(x))(x-a)(x-b)\, dx$$

$$= f''(c) \int_a^b (x-a)(x-b)\, dx$$

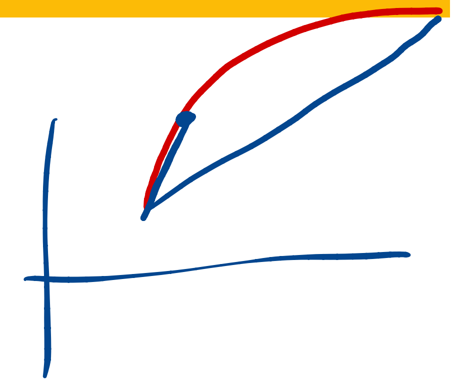
$$\int_a^b \underbrace{(x-a)}(x-b)\, dx = h^2 \int_a^b (1-\theta)\theta\, dx = h^3 \int_0^1 (1-\theta)\theta\, d\theta$$

$$= h^3 \left(\frac{\theta^2}{2} - \frac{\theta^3}{3} \right)\Bigg|_0^1 = \frac{h^3}{6}$$

$$\frac{x-a}{b-a} = \theta$$

Error for Trapezoidal Rule

$$f(x) = g(x) + \frac{f''(\xi)}{2}(x-a)(x-b)$$



$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f''(\xi(x))(x-a)(x-b)\,dx$$
$$= f''(c)\int_a^b (x-a)(x-b)\,dx$$

$$\int_a^b (x-a)(x-b)\,dx = h^2\int_a^b (1-\theta)\theta\,dx = h^3\int_0^1 (1-\theta)\theta\,d\theta$$
$$= h^3\left(\frac{\theta^2}{2} - \frac{\theta^3}{3}\right)\Bigg|_0^1 = \frac{h^3}{6}$$

$$E = |f''(c)|\frac{h^3}{\cancel{6}}$$ 12 ?

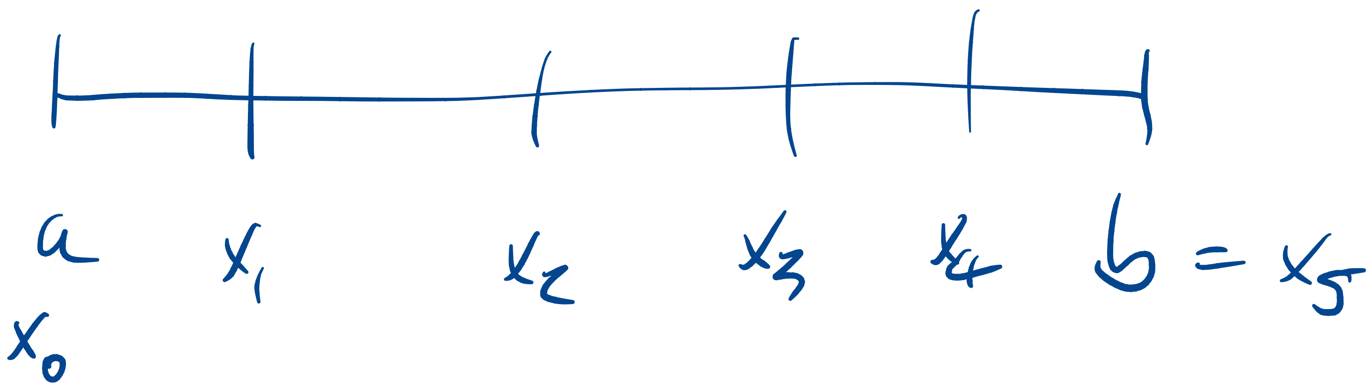
Newton-Coates Rule

Strategy: We subdivide $[a, b]$ into n equally sized intervals and interpolate with a polynomial of degree n. The case $n = 1$ is the trapezoid rule. The case $n = 2$ is known as Simpson's rule.

Newton-Coates Rule

Strategy: We subdivide $[a, b]$ into n equally sized intervals and interpolate with a polynomial of degree n. The case $n = 1$ is the trapezoid rule. The case $n = 2$ is known as Simpson's rule.

Sample points: $a = x_0, x_1, \ldots, x_n = b$.

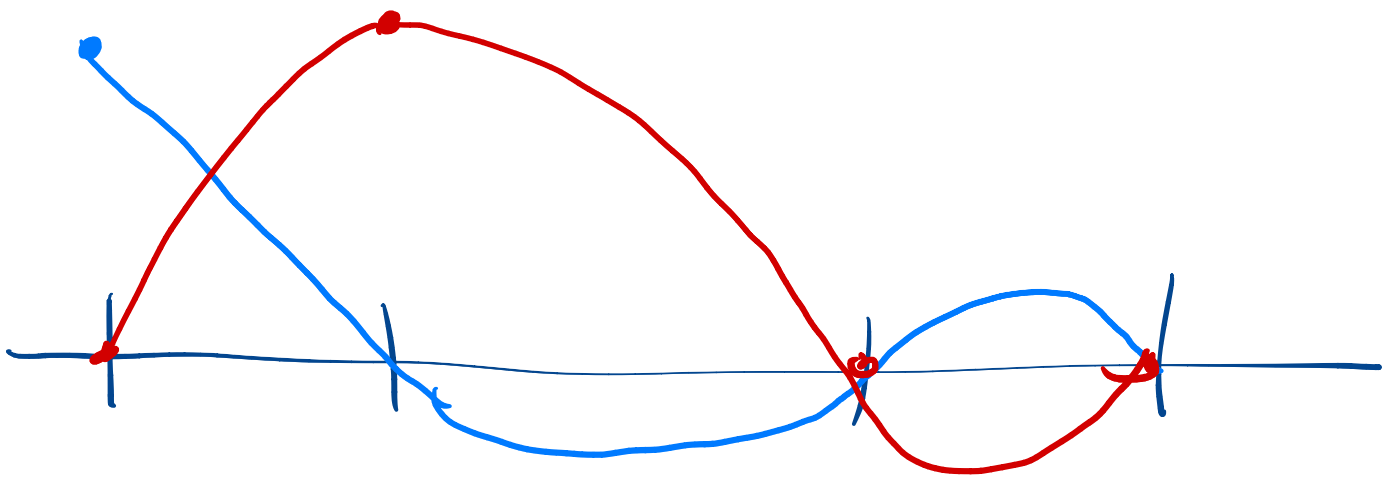


Newton-Coates Rule

Strategy: We subdivide $[a, b]$ into n equally sized intervals and interpolate with a polynomial of degree n. The case $n = 1$ is the trapezoid rule. The case $n = 2$ is known as Simpson's rule.

Sample points: $a = x_0, x_1, \ldots, x_n = b$.

Let p_k be the Lagrange basis polynomial, so $p_k(x_j) = \delta_{k,j}$.



Newton-Coates Rule

Strategy: We subdivide $[a, b]$ into n equally sized intervals and interpolate with a polynomial of degree n. The case $n = 1$ is the trapezoid rule. The case $n = 2$ is known as Simpson's rule.

Sample points: $a = x_0, x_1, \ldots, x_n = b$.

Let p_k be the Lagrange basis polynomial, so $p_k(x_j) = \delta_{k,j}$. Let

$$A_k = \int_a^b p_k(x)\, dx$$

Newton-Coates Rule

Strategy: We subdivide $[a, b]$ into n equally sized intervals and interpolate with a polynomial of degree n. The case $n = 1$ is the trapezoid rule. The case $n = 2$ is known as Simpson's rule.

Sample points: $a = x_0, x_1, \ldots, x_n = b$.

Let p_k be the Lagrange basis polynomial, so $p_k(x_j) = \delta_{k,j}$. Let

$$A_k = \int_a^b p_k(x)\, dx$$

Interpolant:

$$g(x) = f(x_0)p_0(x) + \cdots + f(x_n)p_n(x) = \sum_{k=0}^{n} f(x_k)p_k(x)$$

$$\int g(x)\,dx = \int \sum_{k=0}^{n} f(x_k)p_k(x)\,dx$$

$$\int_a^b g(x)\,dx = \int_a^b \sum_{k=0}^{n} f(x_k)\,p_k(x)\,dx$$

$$= \sum_{k=0}^{n} f(x_k) \underbrace{\int_a^b p_k(x)\,dx}_{A_k}$$

$$= \sum_{k=0}^{n} A_k f(x_k)$$

$$= A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n)$$

$$= A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n)$$

$$h \left(\frac{f(a) + f(b)}{2} \right)$$

$$\frac{h}{2} f(a) + \frac{h}{2} f(b)$$

$$\prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)} = p_k(x)$$

$$\int_a^b p_k(x)\, dx = \int_a^b \qquad dx$$

Newton-Coates Rule

Strategy: We subdivide $[a, b]$ into n equally sized intervals and interpolate with a polynomial of degree n. The case $n = 1$ is the trapezoid rule. The case $n = 2$ is known as Simpson's rule.

Sample points: $a = x_0, x_1, \ldots, x_n = b$.

Let p_k be the Lagrange basis polynomial, so $p_k(x_j) = \delta_{k,j}$. Let

$$A_k = \int_a^b p_k(x)\, dx$$

Interpolant:

$$g(x) = f(x_0)p_0(x) + \cdots + f(x_n)p_n(x) = \sum_{k=0}^{n} f(x_k)p_k(x)$$

$$\begin{aligned}\int_a^b g(x)\, dx &= \int_a^b \sum_{k=0}^{n} f(x_k)p_k(x)\, dx \\ &= \sum_{k=0}^{n} f(x_k) \int_a^b p_k(x)\, dx = \sum_{k=0}^{n} f(x_k)A_k\end{aligned}$$

Newton Coates II

If you know

$$A_k = \int_a^b p_k(x)\, dx$$

then

$$\int_a^b g(x)\, dx = \sum_{k=0}^{n} f(x_k) A_k := Q[f]$$

and

$$\int_a^b f(x)\, dx \approx Q[f].$$

Newton Coates II

If you know

$$A_k = \int_a^b p_k(x)\, dx$$

then

$$\int_a^b g(x)\, dx = \sum_{k=0}^{n} f(x_k) A_k := Q[f]$$

and

$$\int_a^b f(x)\, dx \approx Q[f].$$

How to compute the A_k's without undue pain?

Newton Coates III

How to compute the A_k's without undue pain?

Newton Coates III

How to compute the A_k's without undue pain?

Key observation: the n'th order interpolant of a polynomial of degree n is exact. So if q is any polynomial of degree n or less

$$\boxed{\int_a^b q(x)\,dx} = Q[q] = \sum_{k=0}^{n} q(x_k)A_k.$$

$$\int_a^b q(x)\,dx = \int_a^b g(x)\,dx = \sum_{k=0}^{n} A_k\, q(x_k)$$

Newton Coates III

How to compute the A_k's without undue pain?

Key observation: the n^th order interpolant of a polynomial of degree n is exact. So if q is any polynomial of degree n or less

$$\int_a^b q(x)\,dx = Q[q] = \sum_{k=0}^{n} q(x_k)A_k.$$

Pick your favorite $n+1$ polynomials q_j to obtain $\cancel{n}$ n+1 equations

$$\sum_{k=0}^{n} q_j(x_k)A_k = \int_a^b q_j(x)\,dx.$$

Now solve for the A_k's.

Simpson's Rule

We'll use $q_j(x) = \theta(x)^j$, $j = 0, 1, 2.$

$\theta^0, \theta^1, \theta^2$

$$\int_a^b \theta^j \, dx = h \int_0^1 \theta^j \, d\theta = h \frac{1}{(j+1)}$$

$q_0(x) = \theta^0(x)$
$= 1$

$$A_0 q_0(x_0) + A_1 q_0(x_1) + A_2 q_0(x_2) = h$$

$$A_0 q_1(x_0) + A_1 q_1(x_1) + A_2 q_1(x_2) = \frac{h}{2}$$

$$A_0 q_2(x_0) + A_2 q_1(x_1) + A_2 q_2(x_2) = \frac{h}{3}$$

$$A_0 q_0(x_0) + A_1 q_0(x_1) + A_2 q_0(x_3) = \int_a^b q_0(x) dx$$

$$A_0 + A_1 + A_2 = h$$

Simpson's Rule

We'll use $q_j(x) = \theta(x)^j$, $j = 0, 1, 2.$

$$\int_a^b \theta^j \, dx = h \int_0^1 \theta^j \, d\theta = h \frac{1}{(j+1)}$$

$$A_0 q_0(x_0) + A_1 q_0(x_1) + A_2 q_0(x_2) = h$$

$$A_0 q_1(x_0) + A_1 q_1(x_1) + A_2 q_1(x_2) = \frac{h}{2}$$

$$A_0 q_2(x_0) + A_2 q_1(x_1) + A_2 q_2(x_2) = \frac{h}{3}$$

$$\theta(x_0) = 0; \qquad \theta(x_1) = \frac{1}{2}; \qquad \theta(x_2) = 1$$

Simpson's Rule

We'll use $q_j(x) = \theta(x)^j,\ j = 0, 1, 2.$

$$\int_a^b \theta^j \, dx = h \int_0^1 \theta^j \, d\theta = h \frac{1}{(j+1)}$$

$$A_0 q_0(x_0) + A_1 q_0(x_1) + A_2 q_0(x_2) = h \quad = \int_a^b \theta^0 dx$$

$$A_0 q_1(x_0) + A_1 q_1(x_1) + A_2 q_1(x_2) = \frac{h}{2} \quad \int_a^b \theta^1 dx$$

$$A_0 q_2(x_0) + A_2 q_1(x_1) + A_2 q_2(x_2) = \frac{h}{3} \quad \int_a^b \theta^2 dx$$

$$\theta(x_0) = 0; \qquad \theta(x_1) = \frac{1}{2}; \qquad \theta(x_2) = 1$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 1/4 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} h \\ h/2 \\ h/3 \end{pmatrix}$$

Simpson's Rule

We'll use $q_j(x) = \theta(x)^j, \ j = 0, 1, 2.$

$$\int_a^b \theta^j \, dx = h \int_0^1 \theta^j \, d\theta = h \frac{1}{(j+1)}$$

$$A_0 q_0(x_0) + A_1 q_0(x_1) + A_2 q_0(x_2) = h$$

$$A_0 q_1(x_0) + A_1 q_1(x_1) + A_2 q_1(x_2) = \frac{h}{2}$$

$$A_0 q_2(x_0) + A_2 q_1(x_1) + A_2 q_2(x_2) = \frac{h}{3}$$

$$\theta(x_0) = 0; \qquad \theta(x_1) = \frac{1}{2}; \qquad \theta(x_2) = 1$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 1/4 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} h \\ h/2 \\ h/3 \end{pmatrix}$$

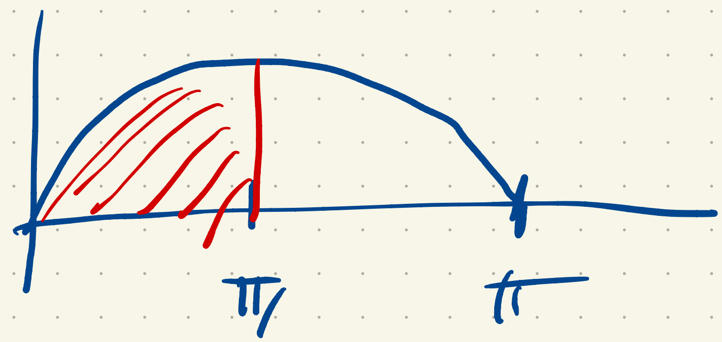
$A_0 = \frac{h}{6}$ $\quad A_2 = \frac{h}{6}$

$A_1 = \frac{2h}{3}$

$$A_0 = \frac{h}{6}; \qquad A_1 = \frac{2h}{3}; \qquad A_2 = \frac{h}{6}$$

$A_0 = \frac{h}{6} \qquad A_1 = \frac{2h}{3} \qquad A_2 = \frac{h}{6}$

$$\int_0^{\pi/2} \underline{\sin(x)}\, dx$$



$$= -\cos(x)\Big|_0^{\pi/2}$$

$$= -\cos(\pi/2) + \cos(0)$$

$$= 1$$