Richardson Extrapolation

Math 426

University of Alaska Fairbanks

December 2, 2020

Motivation

Recall the first order approximation for the derivative:

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{1}{2!}f''(x)h + \frac{1}{3!}f'''(x)h^2 + O(h^3).$$

Motivation

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There is a way to convert a first order approximation into a second order approximation by doing a little more labor.

A bit of abstraction

Rewrite

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{1}{2!}f''(x)h + \frac{1}{3!}f'''(x)h^2 + O(h^3).$$

as

$$F(h) = f'(x) + A_1 h + A_2 h^2 + O(h^3)$$

A_*

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$$F(h) = f'(x) + A_1 h + A_2 h^2 + O(h^3)$$

$$\frac{F(h/2) - \frac{1}{2}F(h)}{1-\frac{1}{2}}$$

Observe

$$F(h/2) = f'(x) + A_1\frac{h}{2} + A_2\left(\frac{h}{2}\right)^2 + O(h^3)$$

and

$$\frac{1}{2}F(h) = \frac{1}{2}f'(x) + A_1\frac{h}{2} + \frac{A_2}{2}h^2 + O(h^3)$$

$$= f'(x) + O(h^2)$$

$$1-\frac{1}{2}$$

So:

$$F(h/2) - \frac{1}{2}F(h) = \frac{1}{2}f'(x) - \frac{A_2}{4}h^2 + O(h^3)$$

The new rule

$$F^{(2)}(h) = \frac{F(h/2) - \frac{1}{2}F(h)}{1 - \frac{1}{2}} = f'(x) - \frac{A_2}{2}h^2 + O(h^3)$$

Matlab Demo

We can improve!

f'(x)

Our new rule has the asymptotic form:

$$F^{(2)}(h) = A_* + A_2^{(2)} h^2 + O(h^3)$$

where $A_* = f'(x)$ is the thing we want to compute.

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Observe:

$$F^{(2)}(h/2) = A_* + \frac{1}{2^2} A_2^{(2)} h^2 + O(h^3).$$

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$$F^{(3)}(h) = \frac{F^{(2)}(h/2) - \frac{1}{2^2} F^{(2)}(h)}{1 - \frac{1}{2^2}} = A_* + O(h^3)$$

We just made an $O(h^3)$ rule.

$$F^{(4)}(h) = \frac{F^{(3)}(h/2) - \frac{1}{2^3} F^{(3)}(h)}{1 - 1/2^3}$$

$A_*\left(1 - \frac{1}{2^2}\right)$

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Observe:

$$F^{(2)}(h/2) = A_* + \frac{1}{2^2} A_2^{(2)} h^2 + O(h^3).$$

so

$$\frac{F^{(2)}(h/2) - \frac{1}{2^2} F^{(2)}(h)}{1 - \frac{1}{2^2}} = A_* + O(h^3)$$

We just made an $O(h^3)$ rule.

You can keep going!

Even Expansions

Recall

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + C_1 h^2 + O(h^4)$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \cdots$$

$$(-h) \quad (-h)^2 \quad (-h)^3$$

$$f(x-h) =$$

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For even expansions you can increase the order by 2 each step.

Integration

The composite trapezoid rule also has an even expansion, but now in terms of n

$$Q^{(2)}(n) = \int_a^b f(x)\, dx + Cn^{-2} + O(n^{-4})$$

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$(2n)^{-2} = 2^{-2} n^{-2}$

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(This is Simpson's rule!)

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