The goal of this worksheet is for you to gain experience with different discrete approximations of the derivative. Throughout we assume that f is a smooth function and x is in its domain.

1. Taylor's Theorem with remainder says

$$f(x+h) = f(x) + f'(x)h + f''(\xi)\frac{h^2}{2}$$

for some ξ between x and $x+h$. Derive a similar formula for $f(x-h)$.

2. Use the previous two formulas to show

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h).$$

3. In fact, this approximation for the derivative does better. First, apply Taylor's theorem with remainder to

$$f(x+h)$$

but now use the second order approximation rather than the first order approximation used in problem 1. Then do the same thing with $f(x-h)$.

4. Use the previous problem to show

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2).$$

5. Is it possible that

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^3)?$$

Try using a third order approximation to $f(x+h)$ and $f(x-h)$ to see what happens.

6. The following formula is another approximation for the first derivative:

$$f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}.$$

To see this, Use first order Taylor approximations to show that

$$\frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = f'(x) + O(h).$$

7. Find the largest value of k such that

$$\frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = f'(x) + O(h^k).$$

To do this, use Taylor approximations of high enough order so that

$$\frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = f'(x) + Ch^k + O(h^{k+1}.$$

for some C that need not be zero.

8. In the previous problems, I gave you the derivative approximation. But you can derive your own rules. Find coefficeints A, B and C such that

$$f''(x) \approx Af(x) + Bf(x+h) + Cf(x+2h)$$

has the highest order accuracy possible. That is, you want

$$Af(x) + Bf(x+h) + Cf(x+2h) = f''(x) + O(h^k)$$

where k is as large as possible. You'll want to replace each of $f(x+h)$ and $f(x+2h)$ with an expression from Taylor's Theorem and determine conditions on A, B and C to solve for them.