Consider the matrix

$$A = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

1. If we were to start Gaussian elimination on B with partial pivoting, we would need to exchange rows 1 and 2. Find a 3×3 exchange matrix E_1 that has the property that for any 3×3 matrix C, $E_1 C$ exchanges rows 1 and 2 and keeps row 3 in place.

2. What is $E_1 \cdot E_1$? Can you predict the answer without doing the matrix multiplication?

3. Let L_0 be the 3×3 identity matrix and let $U_0 = A$. These are your starting approximations for L and U; we'll build them up as we progress. For now, justify the following equation.

$$E_1 A = (E_1 L_0 E_1)(E_1 U_0).$$

4. Let $\hat{U}_0 = E_1 U_0$ and $\hat{L}_0 = E_1 L_0 E_1$.

1. Why is $E_1 A = \hat{L}_0 \hat{U}_0$?
2. Write down $\hat{U}_0$ and $\hat{L}_0$ explicitly.
3. How is $\hat{U}_0$ related to U_0?

It's ok if $\hat{L}_0$ is a little mysterious at this point. At any rate, $\hat{L}_0$ and $\hat{U}_0$ are your new approximations for L and U.

5. Now $\hat{U}_0$ is in good shape for the first round of Gaussian elimination.

1. Clear the first column to compute U_1 and L_1,
2. Verify by multiplying that
$$E_1 A = L_1 U_1.$$

6. If you have computed U_1 correctly, you'll see that the pivot for column 2 lies in row 3. Bummer! Find an exchange matrix E_2 that exchanges rows 2 and 3.

7. Justify the following equation.

$$E_2 E_1 A = (E_2 L_1 E_2) E_2 U_1.$$

8. Compute $\hat{L}_1 = E_2 L_1 E_2$ and $\hat{U}_1 = E_2 U_1$. These are our new approximations to L and U.

9. How is $\hat{U}_1$ related to U_1?

10. This is the really fun and important question. How is $\hat{L}_1$ related to L_1? Make sure you talk to me before progressing past this point.

11. Why is
$$E_2 E_1 A = \hat{L}_1 \hat{U}_1?$$
(Do no hard work; just look at the last two problems).

12. Peform the final round of Gaussian elimination to clear the second column and compute L_2 and U_2.

13. Verify that $E_2 E_1$ is a permutation matrix P.

14. What are P, L and U such that

$$PA = LU?$$

15. Use P, L and U to solve $Ax = b$ where

$$b = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$$