Math F310 Midterm Fall 2020

Name: _____

Student Id: _____

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted. You may use a straightedge if needed.

Turn off anything that might go beep during the exam.

If you need extra space for scratch work, there is a blank page at the end of the exam.

Good luck!

Problem	Possible	Score
1	15	
2	15	
3	10	
4	8	
5	5	
6	12	
7	10	
Total	75	

1. (15 points)

Consider the matrix

$$A = \begin{pmatrix} 1/5 & 0 & 0 \\ 2/5 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

a. [10 points] By hand, perform LU decomposition of A with partial pivoting to find a permuation matrix P and lower and upper triangular matrices L and U such that

$$PA = LU.$$

Use this page and the top of the next for scratch work, and report your answer on the **next** page. You should show enough scratch work so that partial credit can be awarded in the even you make a mistake in your computations.

Continued....

$P =$ $L =$ $U =$

b. [5 points] Given the matrices P, L and U computed in part **[a]**, how would one solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = [1, 2, 3]^T$? Do **not** actually solve the system, just indicate what needs to be done.

2. (15 points)

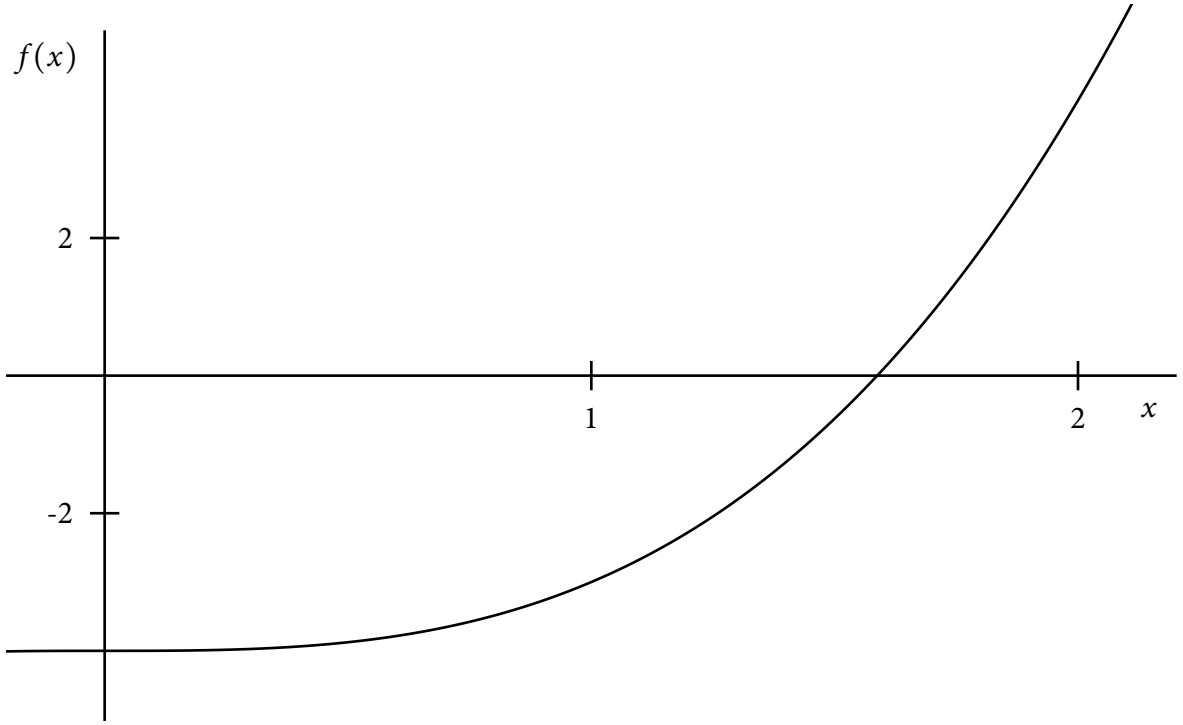
Write a Matlab function `ltrisolve` that implements forwards substitution. It should take a lower-triangular matrix L and a vector $\mathbf{b}$ as its arguments and return a vector $\mathbf{x}$ solving $L\mathbf{x} = \mathbf{b}$. Your code should verify that L is a square matrix, and that $\mathbf{b}$ is a compatible vector.

Note: you may not assume that L has 1's on the diagonal.

3. (10 points)

a. [5 points] Let $f(x) = x^3 - 4$. Apply two iterations of Newton's method starting at $x = 1$ to approximate a solution of $f(x) = 0$.

b. [5 points] The diagram below contains the graph of $f(x)$. In the diagram, make a sketch that illustrates the computation you just completed.



4. (8 points)

Suppose we are performing bisection to solve $f(x) = 0$ on $[100, 200]$ and we known that $f(100) < 0$ and $f(200) > 0$.

a. [2 points] The initial estimate from bisection is that the root is $x_0 = 150$. What is the maximum absolute error for this estimate?

b. [2 points] What is the maximum relative error at x_0?

c. [4 points] Let $x_1, x_2, \ldots$ be the subsequent approximations obtained by bisection. What is the first value of n such that you are guaranteed that the absolute error for x_n is less than 10?

5. (5 points)

Suppose Q is a given $m \times n$ matrix with $m > n$, and $\mathbf{b}$ is a given m-vector. How many floating point operations are required to compute $Q^T\mathbf{b}$? For full credit, you should give a brief justification, in part so that if your answer is in error, partial credit can be assigned.

6. ([12 points])

a. [4 points] State Taylor's Theorem (with the remainder term).

b. [4 points] Compute the third order Taylor polynomial $P(x)$ of $\sin(2x)$ at $x = 0$.

c. [4 points] The third order Taylor polynomial of e^x is $P(x) = 1 + x + x^2/2 + x^3/6$. Estimate the maximum possible value of $|P(x) - e^x|$ for $0 \leq x \leq 2$. You may used the fact that $e^2 < 9$

7. (10 points)

Consider a 10-bit toy IEEE 754 model where a bit pattern

s	e_1	e_2	e_3	e_4	b_1	b_2	b_3	b_4	b_5

represents

$$(-1)^s(1.b_1b_2b_3b_4b_5)_2 \times 2^{(e_1e_2e_3e_4)_2-7}.$$

Aside from the number of bits, all other rules from IEEE 754 apply, including the rules for infinity, not-a-number, and subnormal numbers.

a. [2 points] What is the value of 'machine epsilon' for this scheme?

b. [4 points] What is the largest positive number that can be represented with this system? You **do not** need to perform any arithmetic to answer this question; you may leave your answer as an unsimplified expression involving addition, subtraction, multiplication, and division of honest numbers.

c. [4 points] Let x denote the floating point number in this system that is closest to e, the base of the natural logarithm. Use your answer for part **[a]** to estimate $|x - e|$. Hint: $2 \leq e < 4$.

[BLANK PAGE FOR SCRATCH WORK]