First goal for today: $\mathbb{N}$ is infinite.

Lemma: Suppose $f : S_k \to \mathbb{N}$ for some $k \in \mathbb{N}$. Then $f(S_k)$ is bounded above. (I.e. there exists N in $\mathbb{N}$ such that $N \geq f(j)$ for all $j \in S_k$).

Pf: We proceed by induction on k.
First observe that if $f : S_1 \to \mathbb{N}$ then $f(1)$ is an upper bound for

$f(S_1)$. $\left[f(S_1) = \{f(1)\} \subseteq \mathbb{N} \right]$

$\Downarrow$

$\{1\}$

Suppose for some $k \in \mathbb{N}$ that whenever

$f : S_k \to \mathbb{N}$ then $f(S_k)$ is bounded above.

Now consider some $f : S_{k+1} \to \mathbb{N}$,

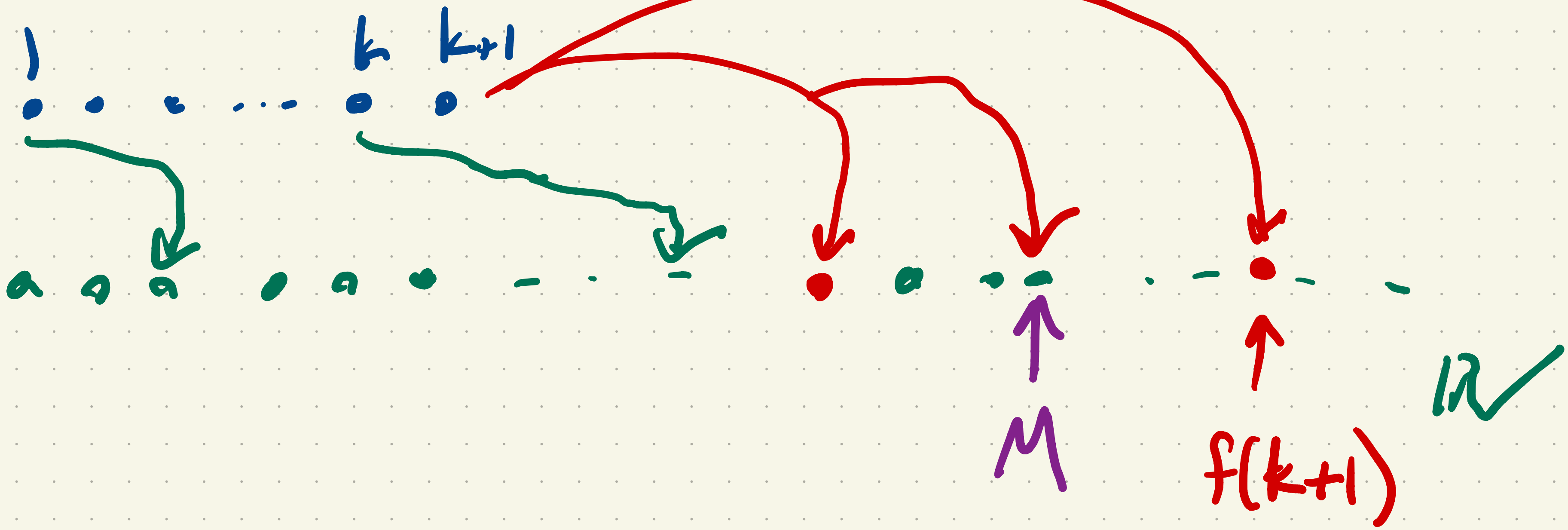
[Job?] [Show $f(S_{k+1})$ is bounded above]

Observe $f(S_{k+1}) = f(S_k) \cup \{f(k+1)\}$.

$\left[S_{k+1} = S_k \cup \{k+1\} \text{ so } f(S_{k+1}) = f(S_k) \cup \{f(k+1)\} \right]$

$$[f(A \cup B) = f(A) \cup f(B)]$$

Let M be an upper bound for $f(S_k)$; M exists by the induction hypothesis.



Observe that $N = \max(M, f(k+1))$ is an upper bound for $f(S_{k+1})$. $\square$

Cor: $\mathbb{N}$ is infinite.

Pf: We will show that for all $k \in \mathbb{N}$ if $f: S_k \to \mathbb{N}$ then f is not surjective.

[$\mathbb{N}$ is infinite $\Leftrightarrow$ $\mathbb{N}$ is not finite

$\Leftrightarrow$

If $\mathbb{N}$ were finite then there would be a bijection $g: S_k \to \mathbb{N}$ for some k.

Consider some $f: S_k \to \mathbb{N}$ for some k.

Let M be an upper bound for $f(S_k)$.

Let $N = M+1$. Then for all $j \in S_k$

$$f(j) \leq M < M+1 = N.$$

Hence $f(j) \neq N$ for all $j \in S_k$ and

therefore f is not sorjective. $\square$

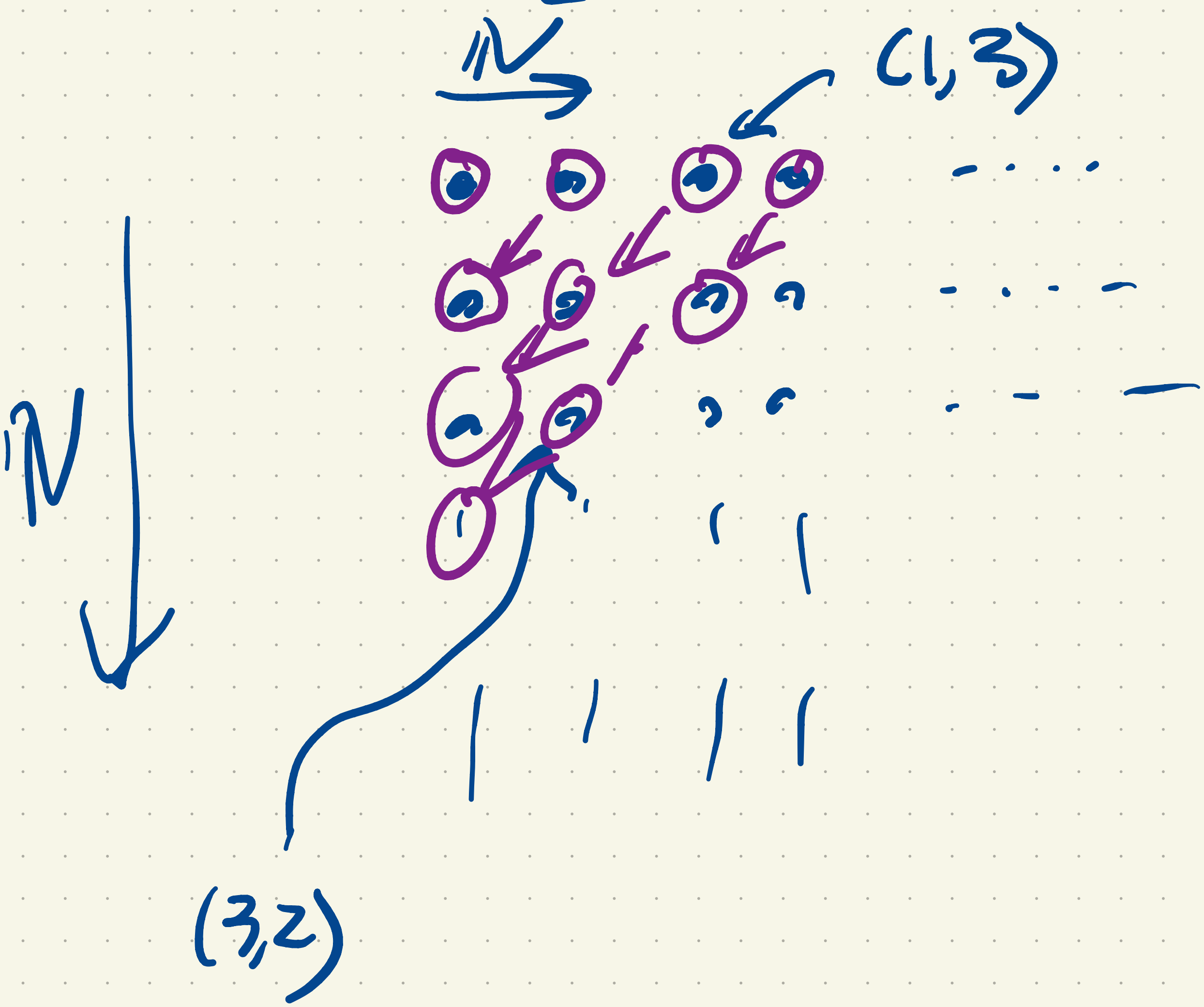
*Prop: If $A \subseteq \mathbb{N}$ then A is at most countable.

*Cor: If $f : \mathbb{N} \to A$ is surjective then A is at most countable.

KEY TOOL

Claim $\mathbb{N} \times \mathbb{N}$ is countably infinite.

$$\mathbb{N} \times \mathbb{N} = \{ (a,b) : a, b \in \mathbb{N} \}$$



Cor: $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$ is countably infinite.

$f(1,2) = \frac{1}{2}$

$f(2,4) = \frac{1}{2}$

Pf: Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_+$ by

$$f(a,b) = a/b.$$

This is evidently surjective.

Let $g : \mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ be a bijection.

Then $f \circ g : \mathbb{N} \to \mathbb{Q}_+$ is a composition of surjections and hence a surjection. $\square$

Exercise: Show from earlier results today that if $A \subseteq B$ and A is infinite then so is B.

$\mathbb{Q}_+$ contains $\mathbb{N}$ which is infinite.

$$\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$$

$\mathbb{Q}_- \to \{q \in \mathbb{Q} : q < 0\}$

On homework: A countably infinite union of at most countable is at most countable.

Cor: A finite union of at most countable sets is at most countable.

$$\underbrace{A_1, \dots, A_k}, \underbrace{\emptyset, \emptyset, \emptyset}$$

Upshot: $\mathbb{Q}$ is countably infinite.