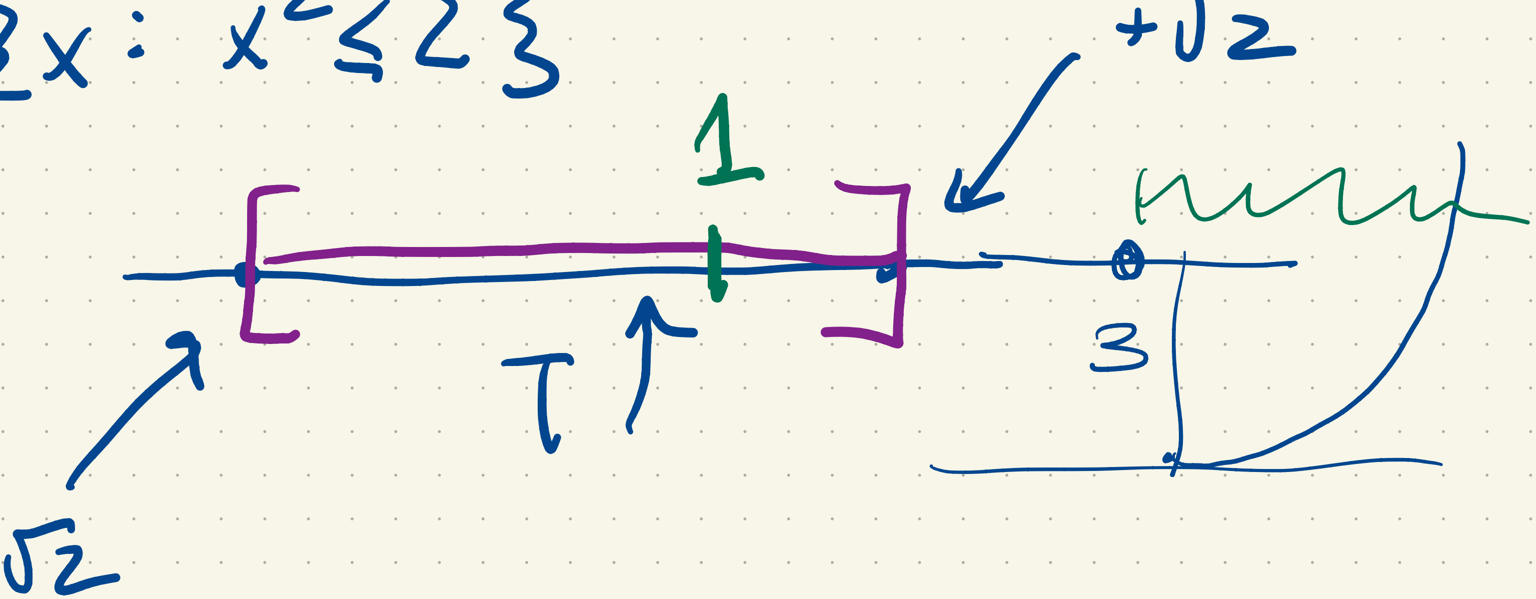
Existence of $\sqrt{2}$

$$T = \{x : x^2 \leq 2\}$$

Mental image



except we haven't shown $\sqrt{2}$ is a thing.

$T \neq \emptyset \quad (0 \in T)$

T bounded above $\left(x, y \geq 0 \quad x \leq y \iff x^2 \leq y^2 \right.$

$\left. \text{3 is an upper bound} \right)$

Let $z = \sup T$.

$z^2 \neq 2$

Claim: $z^2 = 2$.

Suppose to the contrary that $\boxed{z^2 < 2.}$

Let $\varepsilon = 2 - z^2 > 0$.

Pick $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < \frac{\varepsilon}{2}$.

Pick $n_2 \in \mathbb{N}$ such that $\frac{2z}{n_2} < \frac{\varepsilon}{2}$;

i.e. $\frac{1}{n_2} < \frac{\varepsilon}{4z}$. This is possible since

$z \geq 1 > 0$ as $1 \in T$. Let $n = \max(n_1, n_2)$

so $\frac{1}{n} \leq \frac{1}{n_1}$ and $\frac{1}{n} \leq \frac{1}{n_2}$.

Observe $\left(z + \frac{1}{n}\right)^2 = z^2 + \frac{2z}{n} + \frac{1}{n^2}$

$$\leq z^2 + \frac{2z}{n_2} + \frac{1}{n_1^2}$$

$z \quad z^2 < 2$

$\left(z + \frac{1}{n}\right)^2$

$$\leq z^2 + \frac{2z}{n_2} + \frac{1}{n_1}$$

$$< z^2 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= z^2 + \varepsilon$$

$$= 2.$$

Hence $\left(z + \frac{1}{n}\right)^2 < 2$. That is, $\underline{z + \frac{1}{n} \in T}$.

But $z + \frac{1}{n} > z$. This contradicts the fact that z is an upper bound for T.

The other possibility is ruled out by H.W.
$z^2 > 2$ is impossible.

Cardinality (size of sets)

Def: Two sets A, B have the same cardinality if there exists a bijection

$$f: A \to B.$$

Recall: A function is bijective if it is 1-1 and onto
(injective) (surjective)

injective: If $f: A \to B$, f is injective if whenever $f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$.

Identically, if whenever

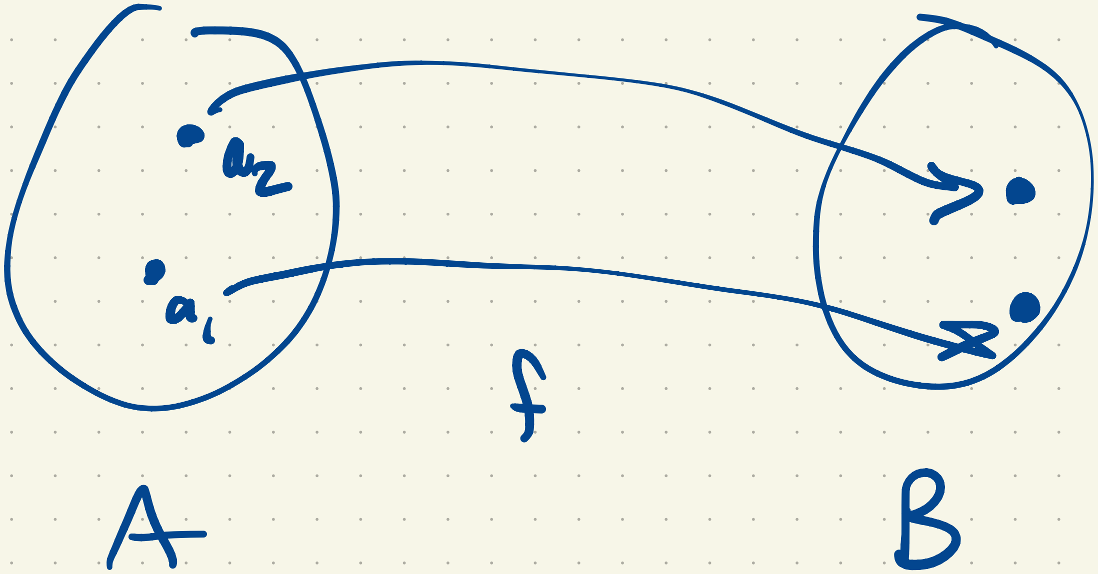
$a_1 \neq a_2$, $f(a_1) \neq f(a_2)$.

surjective $(f: A \to B)$

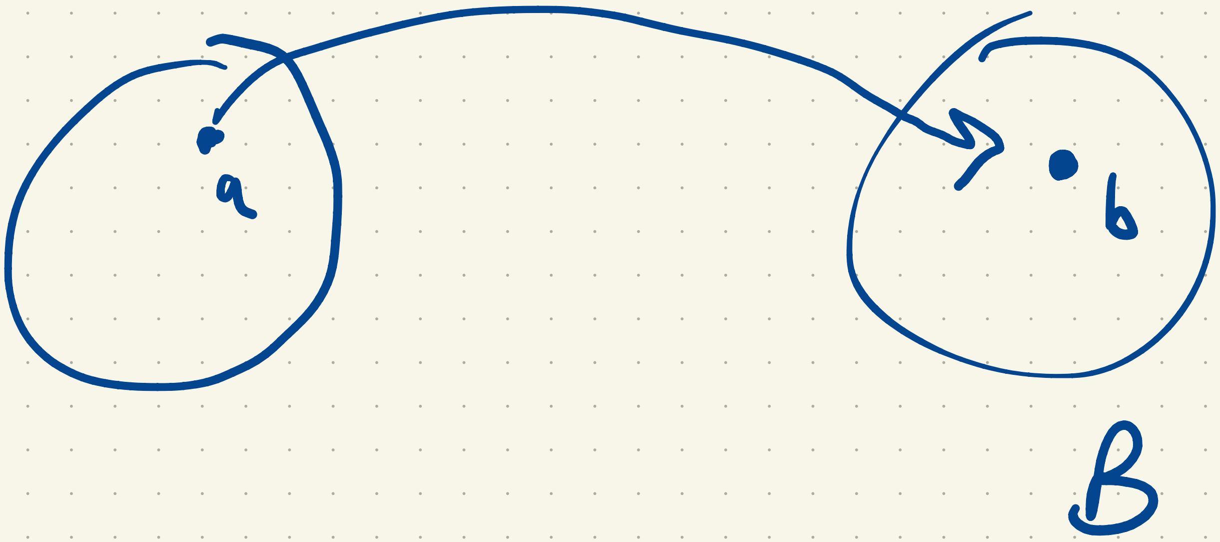
For all $b \in B$ there exists $a \in A$

with $f(a) = b$.

injective

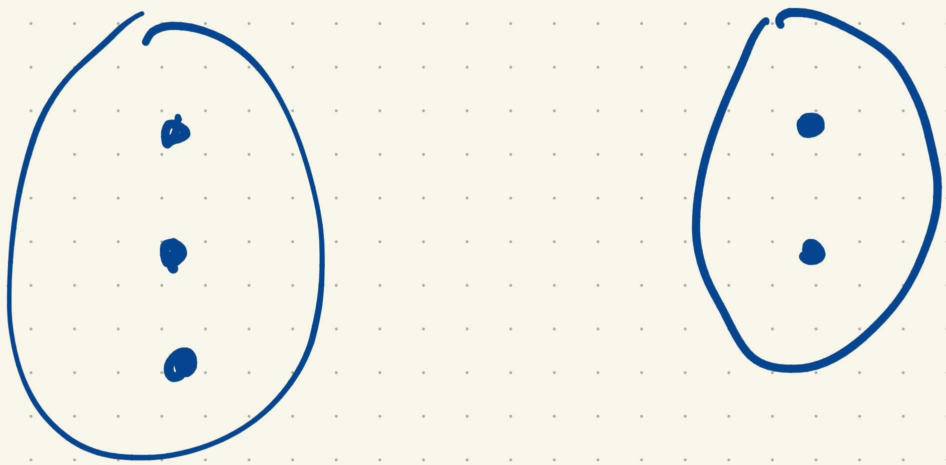


surjective



$$f(A) = B$$

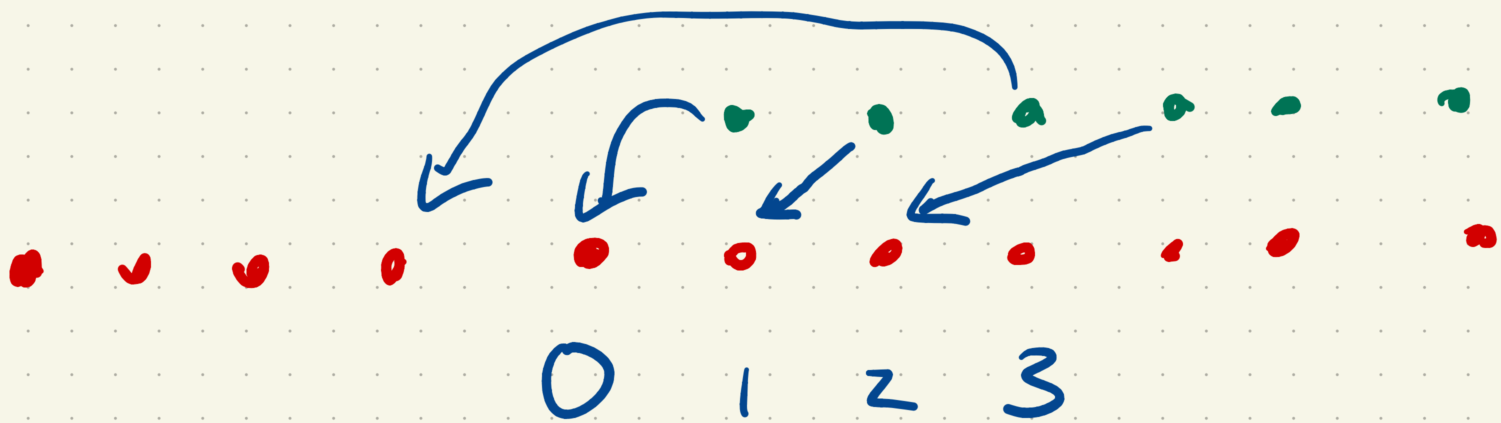
Recall: a function is bijective if and only if it has an inverse function.



These sets have different cardinality.

Goal: $\mathbb{N}$ and $\mathbb{R}$ do not have the same cardinality.

$\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality.



$$f(k) = \begin{cases} k/2 & k \text{ even} \\ \dfrac{1-k}{2} & k \text{ odd} \end{cases} \qquad f : \mathbb{N} \to \mathbb{Z}$$

Exercise: Find an inverse function for f to prove f is a bijection.

We will write $A \sim B$ to mean

A has the cardinality of B.

Exercise: Show that $\sim$ is an equivalence relation between sets.

e.g. $(0,1) \sim (0,\infty)$

$$f(x) = \frac{x}{1+x} \qquad f : \mathbb{R} \to (0,1)$$

$$0 < \frac{x}{1+x} < 1$$

Sizes of sets:

- empty
- finite
- infinite
- countably infinite
- at most countable
- uncountable

Empty: $\emptyset$

Def: For $k \in \mathbb{N}$

$$S_k = \{1, 2, 3, \dots, k\}.$$

Def: A set A is finite if

$$A \sim S_k \text{ for some } k \in \mathbb{N}.$$

If ϕ were finite then there would be a bijection $f: S_k \to \phi$ for some k.

Observe that $1 \in S_k$. But it is impossible for f to assign 1 a value

Def: A set is <u>infinite</u> if it not finite.

We will see, shortly, that $\mathbb{N}$ is infinite

Def: A set is countably infinite if it has the cardinality of $\mathbb{N}$.

Def: A set is at most countable if it is either empty or finite or countably infinite.

Def: A set is uncountable if it is infinity but not countably infinite.

Text: countable = countably infinite

Many others: countable = at most countable