Let $\varepsilon > 0$. Pick N so $100/N < \varepsilon$. $\frac{1}{N} < \frac{\varepsilon}{100}$

$\in \mathbb{N}$

We know that for all n, $a_n - 6 = 32/9n$.

We need to show that if $n \geq N$, $|a_n - 6| < \varepsilon$. Who is ε?

Observe, if $n \geq N$ then

$$|a_n - 6| = \left| \frac{32}{9n} \right| = \frac{32}{9n} \leq \frac{46}{n} \leq \frac{100}{N} < \varepsilon.$$

Who is n?

We need to show that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so if $n \geq N$ then $|a_n - 7| < \varepsilon$. Pick $N \in \mathbb{N}$ so $\frac{1}{N} < \varepsilon$.

Let $\varepsilon > 0$.

Who is ε?

Cor: $\displaystyle\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$ if $|r|<1$.

$\downarrow$

$$\sum_{k=m}^{\infty} r^m r^{k-m} = r^m \sum_{k=0}^{\infty} r^k = \frac{r^m}{1-r}$$

$m=1$ $\qquad \displaystyle\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$

$r = \frac{1}{10}$ $\qquad |r|<1$ $\qquad \displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k = \frac{1/10}{1-1/10} = \frac{1}{10-1}$

$= 1/9$

$\uparrow$

$$\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3}$$

$$0.111\ldots \qquad 0.\overline{1} = \frac{1}{9}$$

$$\sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^k = 9 \sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k = 9 \cdot \frac{1}{9} = 1$$

$$0.999\ldots$$

$$0.\overline{9} = 1 = 1.\overline{0}$$

$5.\overline{9}$

$$5 + \sum_{k=1}^{\infty} \frac{9}{10^k} = 6$$

Cauchy Criterion for sequences:

Sequences converge $\Longleftrightarrow$ Cauchy.

$n > m \geq N$

$(a_n) \Rightarrow$

$n \geq m \geq N$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ so if $n, m \geq N$

$$|a_n - a_m| < \varepsilon.$$

Translate this into the language of series

Cauchy Criterion for Series:

A series $\sum_{k=1}^{\infty} x_k$ converges iff

for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so

if $n > m \geq N$ then

$$\left| \sum_{k=m+1}^{n} x_k \right| < \varepsilon.$$

$s_n = \sum_{k=1}^{n} x_k$ $\quad$ $s_m = \sum_{k=1}^{m} x_k$ $\quad$ $\sum_{k=1}^{\infty} x_k$

$$s_n - s_m = \sum_{k=m+1}^{n} x_k \qquad n > m$$

Cor: If $\sum_{k=1}^{\infty} x_k$ converges then $x_k \to 0$.

[N^{th} term test].

e.g. $\sum_{k=0}^{\infty} r^k$ does not converge if $|r| \geq 1$.

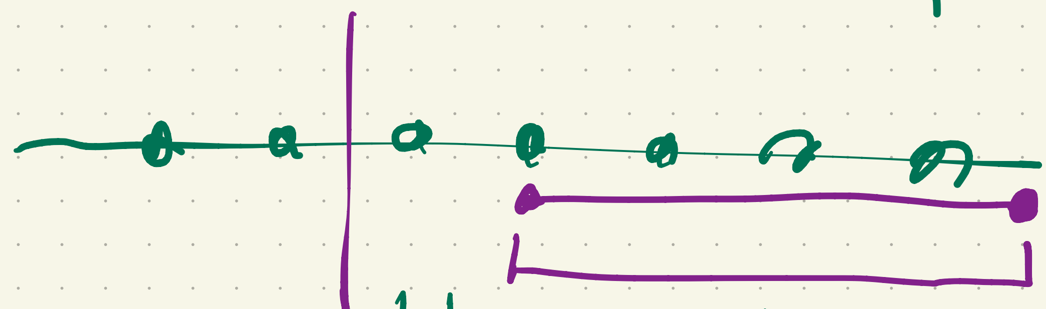
$r^k \not\to 0$. $\quad r > 0$

$r^k \geq r \geq 1$

Pf: Suppose the series converges. Then, Let $\varepsilon > 0$. by the Cauchy Criterion,

There exists $N \in \mathbb{N}$ so if $n > m \geq N$ then

$$\left| \sum_{k=m+1}^{n} x_k \right| < \varepsilon.$$



In particular, if $n > N$ we can take $m = n-1$ to conclude $\left| \sum_{k=n}^{n} x_k \right| < \varepsilon$. That is, if $n > N$,

$$|x_n| < \varepsilon.$$

$\square$

Prop: Consider two series $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$

with $0 \le a_k \le b_k$ for all k.

Then

1) If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

2) If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

$$s_n = \sum_{k=1}^{n} a_k \qquad t_n = \sum_{k=1}^{n} b_k \qquad s_{n+1} \geq s_n$$

$$s_n = a_1 + a_2 + \cdots + a_n; \quad s_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$$

t_n's are increasing and converge

$\Rightarrow$ bounded above

$t_n \le M \quad \forall n.$

$a_1 + a_2 + a_3 \le b_1 + b_2 + b_3$

$s_n \le t_n$

$s_n \le t_n \le M$

$\uparrow$

$s_n \quad t_n$