Last class AND $a \geq 0$

$a, b \in \mathbb{R}$, $a < b$ $\exists\, q \in \mathbb{Q}$

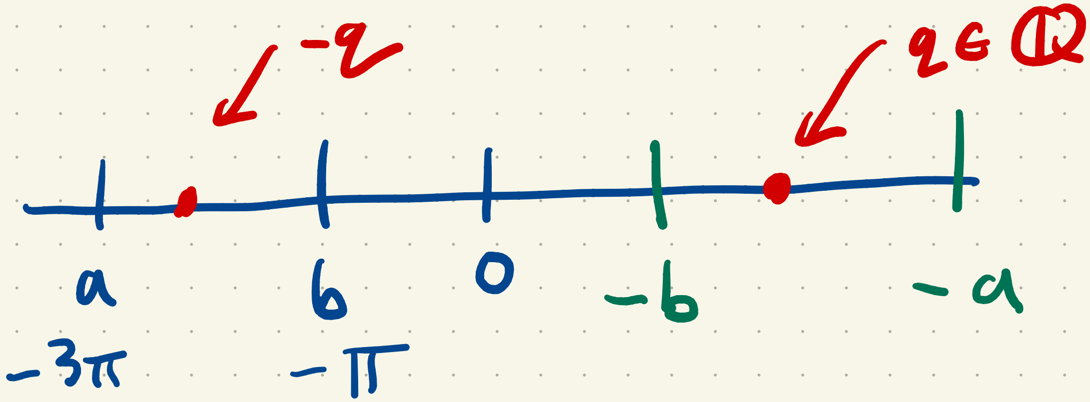
$a < q < b$

$\hookrightarrow$ $a \geq 0$

$a < 0$?

If $b > 0$ $a < 0 < b$ so just use 0.

If $b \leq 0$?



To prove: If $A \subseteq \mathbb{R}$ and $a \in A$ and a is an upper bound for A then $a = \sup A$.

[5/5]

Pf: Suppose $A \subseteq \mathbb{R}$, $a \in A$, and a is an upper bound for A.

- Job: show $a = \sup A$

Explicitly

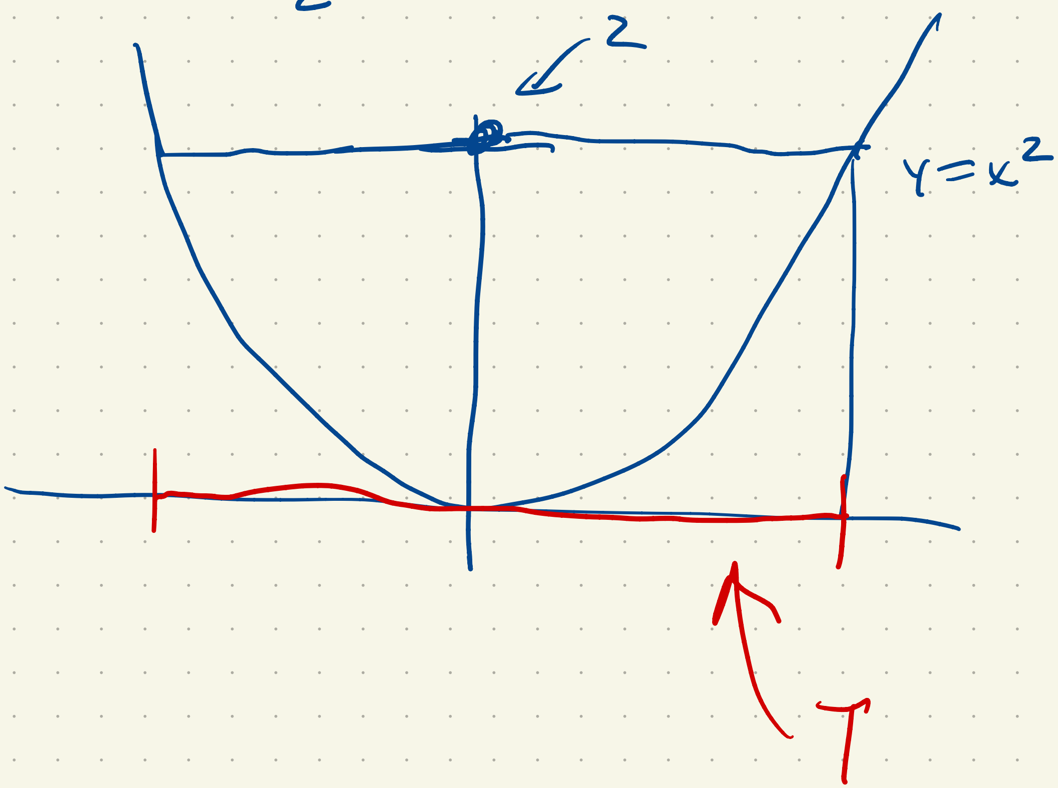
1) Show a is a upper bound for A.

2) Show that if b is an upper bound for A, $a \leq b$.

Existence of square roots:

There exists $x \in \mathbb{R}$ such that $x^2 = 2$.

$$T = \{ x \in \mathbb{R} : x^2 \leq 2 \}$$



candidate is $\sup T$.

We'll show

1) $\sup T$ exists

2) $(\sup T)^2 = 2$

$T = \{x \in \mathbb{R} : x^2 \le 2\}$

Is $T = \emptyset$?

No, because $0 \in T$. (So is 1. So is -1. So 1.1).

Is T bounded above?

Lemma: If $0 \le x \le y$ then $x^2 \le y^2$.

Pf: Exercise.

Assuming this, yes! Consider 3. $3^2 = 9 > 2$

If $y \geq 3$ then $y^2 \geq 3^2 = 9 > 2$.

If $y \geq 3$ then $y^2 > 2$.

If $y \geq 3$ can y be in T?

$$T = \{ x \in \mathbb{R} : x^2 \leq 2 \}$$

If $y \geq 3$ then $y \notin T$.

If $y \in T$ then $y < 3$. $\Rightarrow$ y is an upper bound.

$T \neq \emptyset$, and has an upper bound.

So A.C $\Rightarrow$ T has a supremum, z_0

I claim $z^2 = 2$.

Suppose to produce a contradiction that

$z^2 < 2$.

If we can show that if we increase

z to $z + \frac{1}{n}$ and $(z + \frac{1}{n})^2 < 2$

we have a problem. $z + \frac{1}{n} \in T$.

$z + \frac{1}{n} > z$

z is an upper bound for T

Let $\varepsilon = z^2 - 2$

How big can we increase z?

Exhibit a specific choice of n so that $\left(z + \frac{1}{n}\right)^2 < 2$.