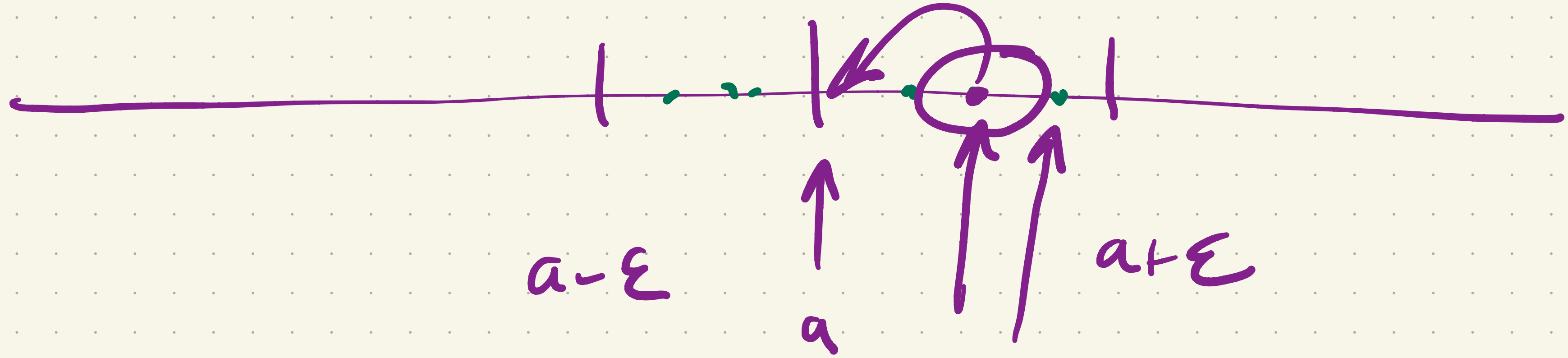


$$A.C \Rightarrow \text{Arch}$$

$$A.C \Rightarrow \text{NIP} + \text{Arch} \Rightarrow \text{BW} \Rightarrow \text{CC}$$

$$A.C \Rightarrow \text{MCT}$$

$$
\begin{tikzcd}
 & \text{NIP} + \text{Arch} \arrow[dl, Leftrightarrow] \arrow[dr, Rightarrow] & & \\
\text{AoC} \arrow[dr, Rightarrow] & & \text{BW} \arrow[dl, Rightarrow] \arrow[r, Leftrightarrow] & \text{CC} + \text{Arch} \\
 & \text{MCT} \arrow[uu, Rightarrow] & &
\end{tikzcd}
$$

$\frac{1}{n} \Rightarrow 0$

MCT $\Rightarrow$ Arch

$\Rightarrow$ NIP

NIP + Arch $\Rightarrow$ AoC

CC + Arch $\Rightarrow$ BW

More on series:

$$\sum_{n=1}^{\infty} a_n$$

$$s_k = \sum_{n=1}^{k} a_n \quad \text{(partial sums)}$$

$$\sum_{n=1}^{\infty} a_n = L \iff s_k \to L$$

If $\sum_{n=1}^{\infty} a_n = A \qquad \sum_{n=1}^{\infty} b_n = B$

$$\underbrace{\sum_{n=1}^{\infty} c a_n}_{\to\, c s_k} = cA$$

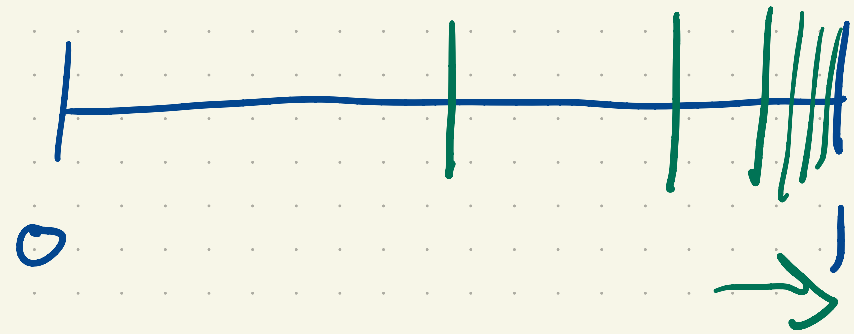
$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

$$\sum_{n=1}^{k} (a_n + b_n) = \underbrace{\sum_{n=1}^{k} a_n}_{\downarrow\, A} + \underbrace{\sum_{n=1}^{k} b_n}_{\downarrow\, B}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$



$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

$$(0.\overline{1})_2 = 1$$

$$\sum_{k=0}^{\infty} 2^{-k} = 2 \qquad \left[\sum_{k=1}^{\infty} 2^{-k} = 1 \right]$$

$$\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)$$

$$= 1 - \frac{1}{16} = 1 - \frac{1}{2^4}$$

$$\left(1 - \frac{1}{2}\right) s_n = \left(1 - \frac{1}{2}\right) \sum_{k=0}^{n} 2^{-k} = 1 - 2^{-(n+1)}$$

$$s_n = \frac{1 - 2^{-(n+1)}}{1 - \frac{1}{2}}$$

$$2^{-n} \to 0$$

$$\lim_{n\to\infty} s_n = \frac{1}{1-\frac{1}{2}} = 2$$

$$\sum_{k=0}^{\infty} r^k \qquad \left(r = \frac{1}{2}\right)$$

$\hookrightarrow$ geometric series

Key: $\lim_{k\to\infty} 2^{-k} = 0$ $\quad 0 < \frac{1}{2^k} \leq \frac{1}{k}$ $\quad k \in \mathbb{N}$

Lemma: Suppose $0 < r < 1$. Then $\lim_{k\to\infty} r^k = 0$.

Pf: Observe that the sequence r^k is monotone decreasing:

$$r^{k+1} = r\, r^k < 1 \cdot r^k = r^k.$$

Note also that $0 < r^k$ for all $k \in \mathbb{N}$.

So the MCT implies $r^k \to \ell$ for some $\ell \geq 0$.

Consider the subsequence $y_k = r^{2k}$.

So $y_k \to \ell$ as well. But $y_k = r^{2k} = r^k \cdot r^k$.

The ALT implies $r^k \cdot r^k \to \ell \cdot \ell$. That is

$y_k \to \ell$ and $y_k \to \ell^2$. Hence $\ell = \ell^2$,

$$\left[\ \ell = \ell^2 \iff \ell^2 - \ell = 0 \iff \ell(1-\ell) = 0\ \right]$$

and $\ell = 0$ or $\ell = 1$. But the ~~series~~

sequence is decreasing and $r^1 < 1$.

So $\ell \le r^1 < 1$ as well. So $\ell = 0$.

$\square$

HW: If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow a_n \to 0.$

Exercise: If $|r|<1$ then $\lim_{k\to\infty} r^k = 0$

Prop: If $|r|<1$ then

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

Pf: Let $s_n = \sum_{k=0}^{n} r^k$. ~~Observe~~ A proof by induction shows

$$(1-r)s_n = r^0 - r^{n+1} = 1 - r^{n+1}.$$

But then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \dfrac{1-r^{n+1}}{1-r}$ previous lemma

$$= \frac{1-0}{1-r} = \frac{1}{1-r}. \quad \square$$

Cor: $\displaystyle\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$ if $|r|<1$.

$0.\overline{9}$ $\displaystyle\sum_{k=1}^{\infty} \frac{9}{10^k} = 9 \sum_{k=1}^{\infty} \Big(\underbrace{\tfrac{1}{10}}_{r}\Big)^k$

$$= 9 \cdot \left[\frac{1/10}{1-1/10}\right] = 9 \cdot \left[\frac{1}{10-1}\right]$$