10:15

10:17

Thm (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence

Pf: Let $\{x_n\}$ be a bounded sequence and pick $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all n.

We first show that we can build nested

closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

such that $|I_k| = 4M2^{-k}$ and such that I_k contains infinitely many terms of the sequence.

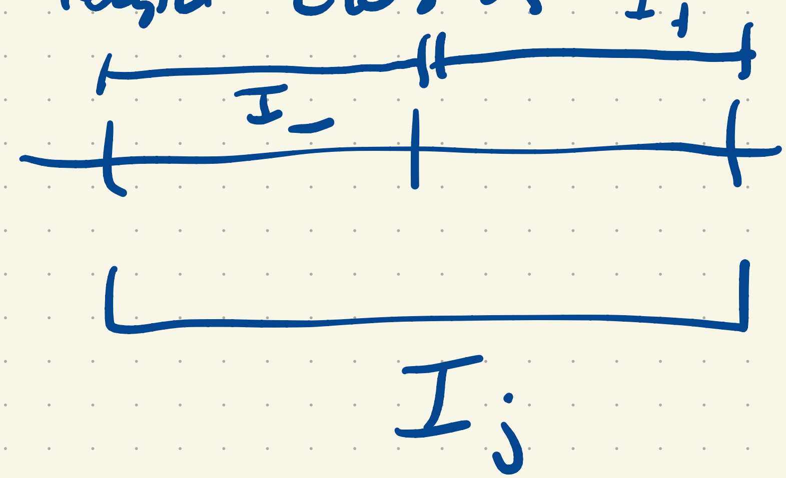
Let $I_1 = [-M, M]$, and note that $|I_1| = 2M$

$= 4M \cdot 2^{-1}$

and that I_1 contains the entire sequence.

Now suppose $I_1, \ldots, I_j$ have been constructed with the desired properties.

Divide I_j into two equal length closed subintervals I_+ and I_-.



Observe that

$$|I_+| = |I_-| = |I_j|/2$$

$$= 4M \cdot 2^{-j}/2 = 4M \cdot 2^{-(j+1)}.$$

Moreover one of I_+ or I_- must contain infinitely many terms of the sequence since I_j does.

From the NIP we know that there exists

some $x \in \bigcap I_j$.

Let $n_1 = 1$ so $x_{n_1} \in I_1$.

Pick n_2 to be the least integer such that a) $n_2 > n_1$

b) $x_{n_2} \in I_2$.

Continuing inductively we can pick indices n_k such that $n_{k+1} > n_k$ and

$x_{n_k} \in I_k$.

I claim $x_{n_k} \to x$.

Let $\varepsilon > 0$. Pick $K \in \mathbb{N}$ such that

$$2M2^{-K} < \varepsilon.$$

$2^{-K} < \frac{\varepsilon}{2M}$

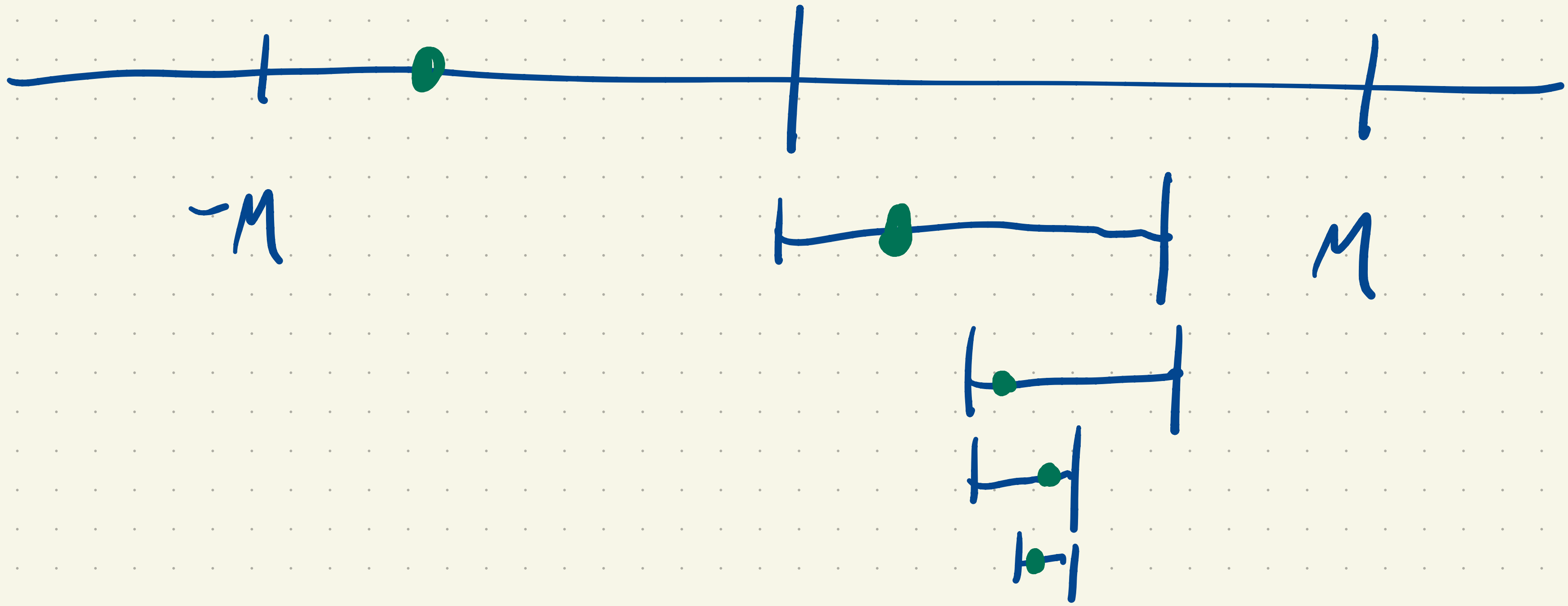
$2^{-k} \to 0$

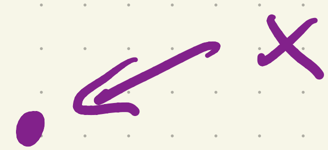
Exercise! $2^k \geq k \ \forall k \in \mathbb{N}$

Then if $k \geq K$ $x_{n_k}, x \in I_k$

so $|x - x_{n_k}| \leq |I_k| = 2M2^{-k} \leq 2M2^{-K} < \varepsilon.$ $\square$

$$0 \leq 2^{-k} \leq \frac{1}{k}$$





When does a sequence converge?

a) Not bounded $\Rightarrow$ does not converge

b) Monotone + bounded $\Rightarrow$ converges

c) Bounded ⇒ a subsequence converges

d) If the terms are settling closer + closer together, the sequence converges

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$\lim_{n \to \infty} a_n = a$ "the terms a_n get closer + closer to a"

Formally: Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n \geq N$, $|a_n - a| < \varepsilon$.

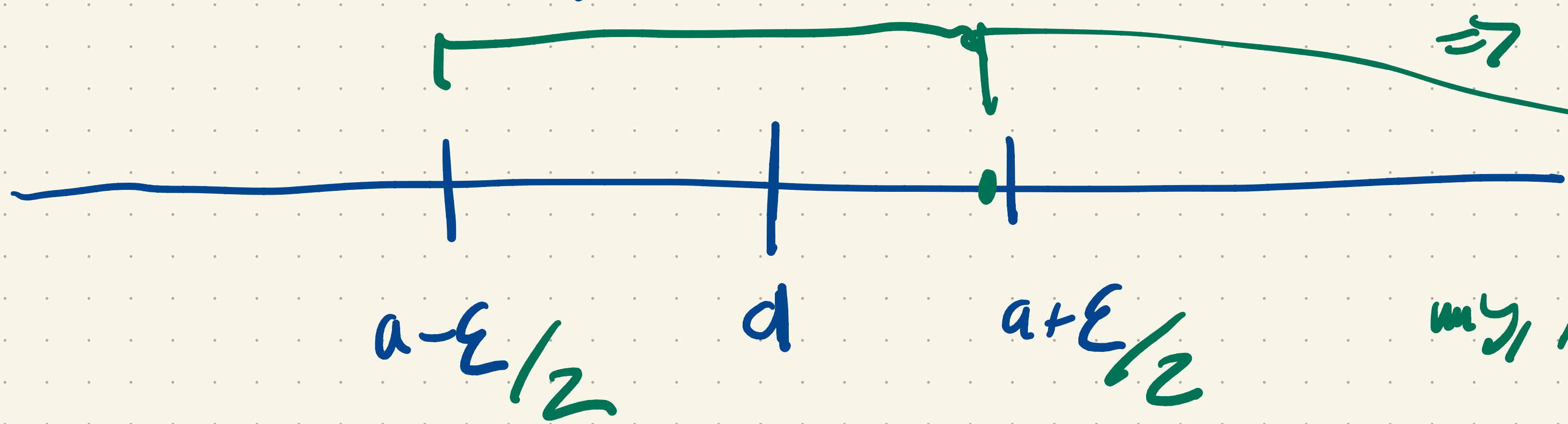
a_n's get closer + closer together?

Def: We say a sequence (a_n) is <u>Cauchy</u> if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|a_n - a_m| < \varepsilon$.

Are convergent sequences Cauchy?

N

$n \geq N \Rightarrow a_n \in$



$m \geq N$

$a_m \in$

$|a_n - a_m| <$

Goal: Cauchy sequences converge!

First step: Cauchy sequences are bounded.

Pf: Suppose (a_n) is a Cauchy sequence.

Pick $N \in \mathbb{N}$ such that if $n, m \geq N$,

$|a_n - a_m| < 1$.

Let $M = \max(|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1)$.

If $n < N$ then clearly $|a_n| \leq M$.

Moreover, if $n \geq N$ then

$$
\begin{aligned}
|a_n| &= |a_n - a_N + a_N| \\
&\leq |a_n - a_N| + |a_N| \\
&< 1 + |a_N| \\
&\leq M.
\end{aligned}
$$

Goal: Cauchy sequences converge.

Cauchy $\Rightarrow$ bounded $\Rightarrow$ has a convergent subsequence

Thm (Cauchy Criterion)

A sequence converges if and only if it is Cauchy.

Pf: We have already seen that convergent sequences are Cauchy.

Let (a_n) be a Cauchy sequence.

Is is bounded and so by BW it has a convergent subsequence (a_{n_k}) converging to some limit a.

Let $\varepsilon > 0$.

Pick N so that if $n, m \geq N$ then

$|a_n - a_m| < \varepsilon/2$.

$n_K \geq K$

Pick $K \in \mathbb{N}$ such that if $k \geq K$

then $|a - a_{n_k}| < \varepsilon/2$. Without loss

of generality we can assume $K \geq N$.

As a consequence, $n_K \geq K \geq N$

as well. ($n_k \geq k$)

Now if $n \geq N$,

$$
\begin{aligned}
|a - a_n| &= |a - a_{n_K} + a_{n_K} - a_n| \\
&\leq |a - a_{n_K}| + |a_{n_K} - a_n| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon .
\end{aligned}
$$

Cauchy

$\square$

Convergence