$S_n \to \pi^2/6$

$\sum_{n=1}^{\infty} a_n$

$s_k = \sum_{n=1}^{k} a_n$

partial sums

$\sum_{n=1}^{\infty} \frac{1}{n}$ [Harmonic series]

$s_1 = 1$; $s_2 = 1.5$; $s_3 = 1 + \frac{1}{2} + \frac{1}{3} = 1\,5/6$

S_{10}	2.92	S_{500}	6.79
S_{20}	3.59	S_{1000}	7.49
S_{50}	4.4992	S_{10000}	9.78
S_{100}	5.19	S_{100000}	12...
S_{200}	5.88		

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq \frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq \frac{1}{8}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{15}+\frac{1}{16}}_{\geq \frac{1}{16}}$$

$$2 \qquad\qquad 4 \qquad\qquad 8$$

$$\geq \frac{1}{2} \qquad\qquad \geq \frac{1}{2} \qquad\qquad \geq \frac{1}{2}$$

Thus series does not converge.

The partial sums are not bounded

$$\boxed{S_{2^k} \geq \frac{k}{2}} \quad \forall k \in \mathbb{N}$$

pf is by induction

Subsequences

Consider $\{x_n\}_{n=1}^{\infty}$

$(x_1, x_2, \boxed{x_3}, x_4, \boxed{x_5}, x_6, x_7, \boxed{x_8}, \ldots\ldots)$

Look at a sequence $\{n_k\}_{k=1}^{\infty}$ of indices.

$(3, 5, 8, \ldots\ldots)$

$(n_1, n_2, n_3, \ldots\ldots)$

$\boxed{(x_j)_{j=-19}^{\infty}}$

$$\{n_k\}_{k=1}^{\infty} \qquad n_k \in \mathbb{N}$$

$$n_{k+1} > n_k \quad \forall k \in \mathbb{N}.$$

$$\{x_{n_k}\}_{k=1}^{\infty} \leftarrow \text{subsequence}$$

Lemma: Suppose $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers with the property

$$n_{j+1} > n_j \quad \text{for all } j \in \mathbb{N}.$$

Then for each $j \in \mathbb{N}$, $n_j \geq j$.

Pf: Observe $n_1 \geq 1$ since $n_1 \in \mathbb{N}$.

Suppose for some $j \in \mathbb{N}$, $n_j \geq j$.

[Job: Show $n_{j+1} \geq j+1$]

Then $n_{j+1} > n_j \geq j$.

Since $n_{j+1} \in \mathbb{N}$ and $n_{j+1} > j$ it follows that $n_{j+1} \geq j+1$. $\square$

Prop: A subsequence of a convergent sequence converges to the same limit.

Pf: Let $(a_k)_{k=1}^{\infty}$ be a sequence converging to some a. Let $(a_{k_j})_{j=1}^{\infty}$ be a subsequence. [Job: Show $a_{k_j} \to a$.]

Let $\varepsilon > 0$. [Job: Show there is J that works.]

Since $a_k \to a$ there exists K so if

$k \geq K$, $|a_k - a| < \varepsilon$.

Let $J = K$. If $j \geq J$ then

$k_j \geq j \geq J = K$.

So, if $j \geq J$,

$$\left| a - a_{k_j} \right| < \varepsilon.$$

$\square$

ε

K

$$(a_1, a_2, a_3, a_4, a_5, \ldots \text{———})$$

$$(a_{k_1}, a_{k_2}, a_{k_3}, \ldots \text{———})$$

Cor: If (a_k) is a sequence with two subsequences that converge to different limits, the sequence does not converge.

E.g. $(-1)^n$ does not converge.

(a_n)

$n_k = 2k$

$(a_{2k})_{k=1}^{\infty}$

$\downarrow$

$(-1)^{2k} = 1$

$(a_{2k+1})_{k=1}^{\infty}$

$\to -1$

Bounded sequences need not converge.

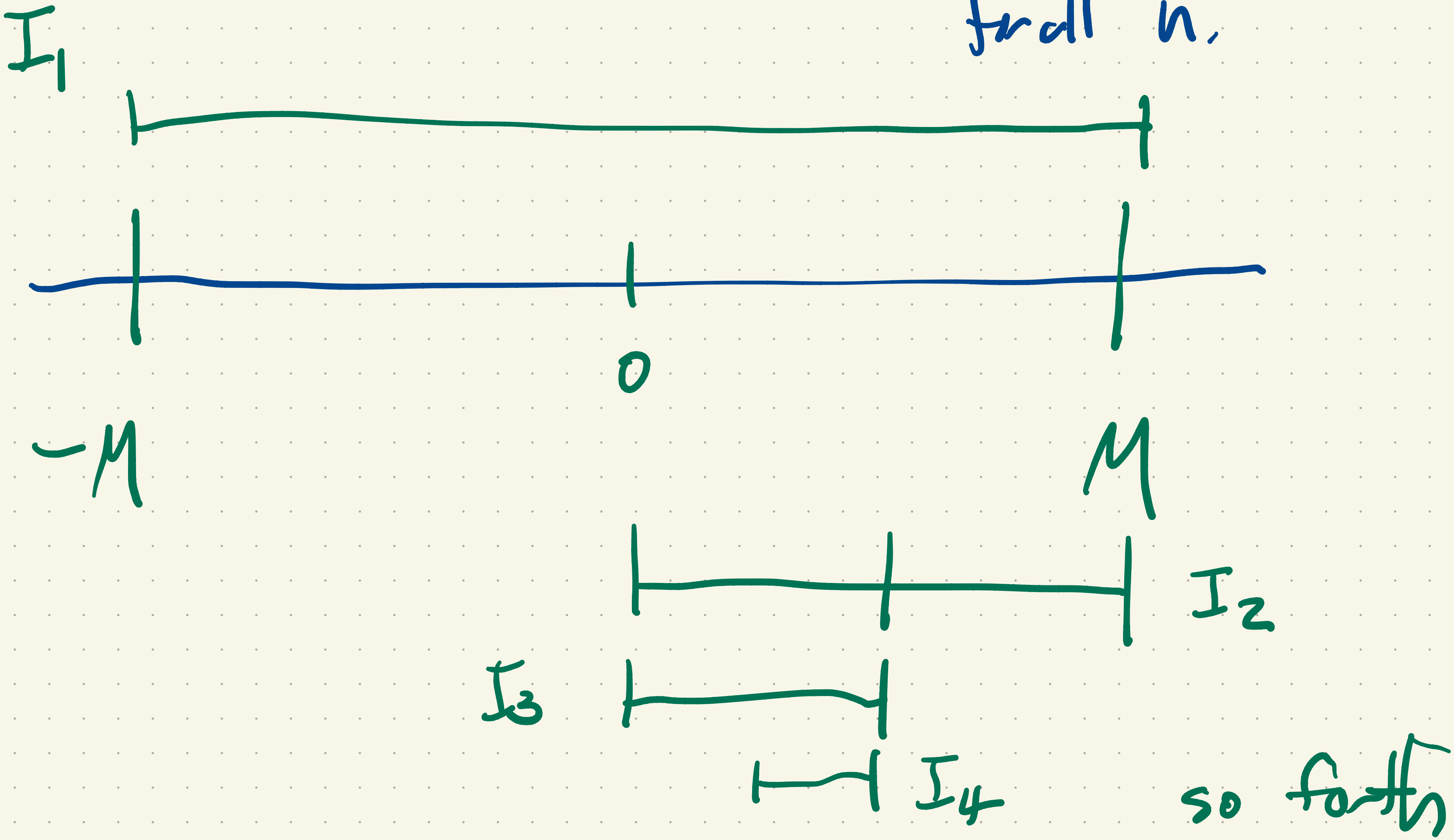
Monotone bounded sequences converge.

Convergent sequences are bounded.

Still bounded sequences are almost good enough to be convergent sequences.
They contain convergent subsequences.

(a_n), bounded.

$|a_n| \le M$ for some M,
for all n.



I_k: nested, closed

$|I_k|$ ← length of interval.

$4M/2^k$

$|I_1| = 2M$

$|I_2| = 2M/2$

$|I_3| = 2M/2^2$

$|I_4| = 2M/2^3$

$|I_k| = 2M/2^{k-1}$

$= M2^{1-k}$

$\varepsilon \quad J$

$\cancel{2}4M/2^J < \varepsilon$

$$\bigcap_{k=1}^{\infty} I_k \neq \emptyset$$

$$x \in \bigcap_{k=1}^{\infty} I_k$$

show there is a subsequence converging to x.

(x_k) build a subsequence

$k_1 = 1$

I_2 contains infinitely many terms.

k_2 least k so $k_2 > k_1$ $k_2 \in I_2$

I_3

k_3 least k so $k_3 > k_2$

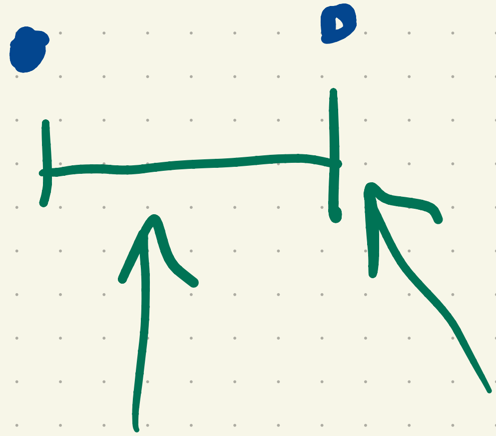
$k_3 \in I_3$

$k_1 < k_2 < k_3 \ldots \ldots$

$x_{k_j} \in I_j$

$x_{k_j} \in I_{\hat{j}}$ so long as $j \geq \hat{j}$

Job: $x_{k_j} \longrightarrow x$



I_J

$x \in I_J$

$x_{k_j} \in I_J$ if $j \geq J$

$$|x - x_{k_j}| \leq 2M/2^J < \varepsilon$$