Last class:

Monotone Convergence Thm.

(x_n), monotone increasing and bounded above

$\Rightarrow$ convergence $\left[x_n \longrightarrow \sup\{x_n : n \in \mathbb{N}\} \right]$

If (x_n) is monotone and bounded then it converges.

$|x_n| \leq M$ $\quad -M \leq x_n \leq M$

Monotone decreasing: (x_n) $\quad x_{n+1} \leq x_n \quad \forall n \in \mathbb{N}$

increasing $\quad x_{n+1} \geq x_n$

Monotone: monotone inc or dec

A sequence (x_n) is bounded below if

there exists $m \in \mathbb{R}$ such that

$$m \leq x_n \quad \forall\, n \in \mathbb{N}.$$

Exercise: A sequence is bounded iff it is bounded above and bounded below.

Claim: A monotone increasing sequence is bounded below.

(x_n) monotone inc.

x_1 is a lower bound.

$x_1 \leq x_1$

If $x_1 \leq x_n$ then $x_1 \leq x_n \leq x_{n+1}$

Exercise: Show that if (x_n) is monotone decreasing then $-x_n$ is monotone increasing and use this to establish the MCT for decreasing sequences

$d_0 . d_1 d_2 d_3 d_4 \ldots$

E.g: Consider a sequence

$$\{ d_k \}_{k=0}^{\infty}$$

where each $d_k \in \{0, 1, 2, \ldots, 9\}$.

$$x_n = \sum_{k=0}^{n} \frac{d_k}{10^k}$$

$$x_{n+1} = x_n + \frac{d_{n+1}}{10^{n+1}}$$

$$x_{n+1} \geq x_n$$

e.g.

$d_0 = 3$

$d_1 = 1$

$d_2 = 4$

$\vdots$

$x_0 = 3$

$x_1 = 3 + \frac{1}{10} = 3.1$

$x_2 = 3 + \frac{1}{10} + \frac{4}{100} = 3.14$

$\vdots$

The x_n's are monotone increasing.

To show the x_n's converge it is enough to show that the sequence is bounded above.

$d_0 = 9$

$x_0 \leq 9$ $\quad$ $d_1 = 7$ $\quad$ $x_2 = 9 + \frac{7}{10} = 9.7$

$x_1 \leq 9.9$ $\quad$ $\forall k \in \mathbb{N}$

$x_2 \leq 9.99$ $\quad$ $x_k \leq 10 - 10^{-k}$

$x_0 \leq 10 - 1 = 9$

$x_1 \leq 10 - 10^{-1} = 9.9$

$$x_k \le \underbrace{10 - 10^{-k}}_{\le 10} \quad \forall k$$

The sequence is bounded above by 10

The x_n's converge to some limit.

A series: $\displaystyle\sum_{n=1}^{\infty} a_n$ $\qquad a_n \in \mathbb{R}$

Partial sums: $\displaystyle s_k = \sum_{n=1}^{k} a_n$

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4 = \sum_{n=1}^{4} a_n$$

We say a series converges if its partial sums converge. Otherwise we say it diverges.

E.g. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{k=1}^{\infty} a_k$

If $a_k \geq 0$

$\Rightarrow s_k$ monotone increasing.

$$\left[\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{4} \\ s_3 &= 1 + \frac{1}{4} + \frac{1}{9} \end{aligned}\right.$$

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)} \qquad n \geq 2$$

$$= \frac{1}{n-1} - \frac{1}{n}$$

$$\frac{n-(n-1)}{n(n-1)} = \frac{1}{n(n-1)}$$

$$\sum_{k=1}^{\infty} \frac{1}{n^2}$$

$$\underline{\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}}$$

$$1 + \frac{1}{2^2} \leq 1 + \left(1 - \frac{1}{2}\right) \leq 2 - \frac{1}{2}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} \leq 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \leq 2 - \frac{1}{3}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \leq 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\leq 2 - \frac{1}{4}$$

$$\boxed{s_n \leq 2 - \frac{1}{n}} \quad \forall n \qquad s_n \leq 2 \quad \forall n.$$

$$S_n \to \pi^2/6$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

[Harmonic series]

S_{10}	2.92	S_{500}	6.79
S_{20}	3.59	S_{1000}	7.49
S_{50}	4.4992	S_{10000}	9.78
S_{100}	5.19	S_{100000}	12. . .
S_{200}	5.88		

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq \frac{1}{4}\cdot 2\ \geq\ \frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq \frac{1}{8}\cdot 4\ \geq\ \frac{1}{2}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{15}+\frac{1}{16}}_{\geq \frac{1}{16}\cdot 8\ \geq\ \frac{1}{2}}$$

This series does not converge.

The partial sums are not bounded

$$\boxed{S_{2^k} \geq \frac{k}{2}} \quad \forall k \in \mathbb{N}$$

pf is by induction