Then for all $n \in \mathbb{N}$, $\frac{1}{|b_n|} \leq M$. $\square$

Prop: Suppose $b_n \neq 0$ $\forall n \in \mathbb{N}$, and

$b_n \to b \neq 0$. Then $1/b_n \to 1/b$.

Pf: Let $M > 0$ be a bound such that

$|1/b_n| \leq M$ for all n; this bound

exists because of the previous lemma.

Let $\varepsilon > 0$. [Job: Find an N that works.]

Since $b_n \to b$ there exists $N \in \mathbb{N}$ such

that $|b_n - b| < \frac{\varepsilon |b|}{M} \cancel{M}$ for all $n \geq N$.

Then, if $n \geq N$,

$$\left| \frac{1}{b} - \frac{1}{b_n} \right| = \frac{|b_n - b|}{|b|\,|b_n|} \Rightarrow |b_n - b| \cdot \frac{1}{|b|} \cdot \frac{1}{|b_n|}$$

$$\leq |b_n - b| \frac{1}{|b|} \cdot M$$

$$< \varepsilon \frac{|b|}{M} \cdot \frac{M}{|b|} = \varepsilon. \qquad \square$$

$a_n \to a \qquad b_n \to b$

i) $a_n + b_n \to a + b$

ii) $a_n b_n \to ab$

iii) $1/b_n \to 1/b \qquad \left(\begin{array}{l} b \neq 0 \\ b_n \neq 0 \ \forall n \end{array} \right)$

[Exercise: If $b \neq 0$ then there is N so if $n \geq N$, $b_n \neq 0$.]

Cor: iv) $c a_n \to ca \quad \forall c \in \mathbb{R}$.

$b_n = c \ \forall n$

v) $a_n - b_n \to a - b$

vi) $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ $\left(\begin{array}{l} b \neq 0 \\ b_n \neq 0 \quad \forall n \end{array} \right)$

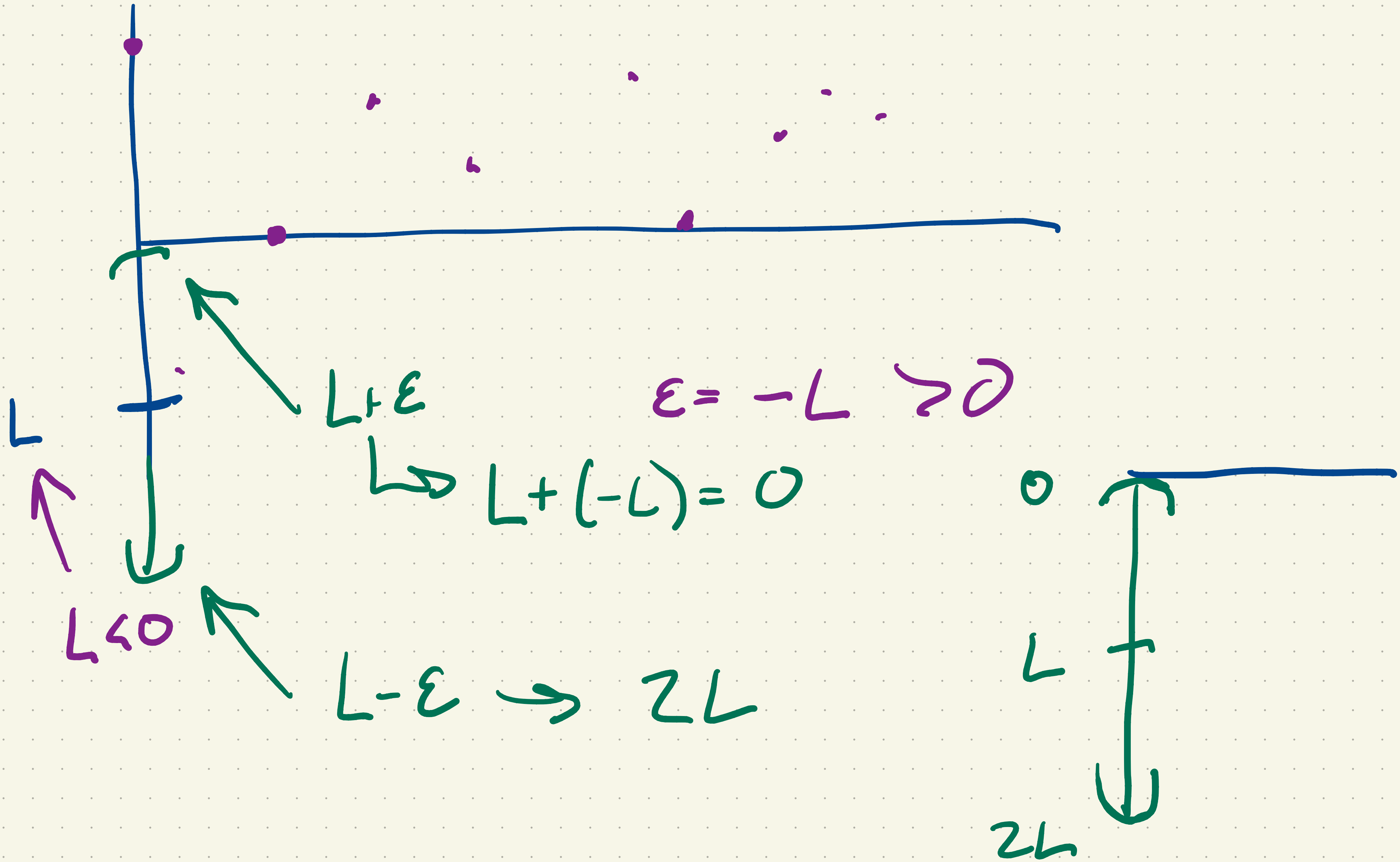
Limits and Order:

$$\boxed{a_n \geq 0 \qquad a_n \to L \geq 0}$$

$a_n > 0 \qquad a_n \to L \geq 0$

$> ?$

$a_n = \frac{1}{n} \quad a_n \to 0 \quad a_n > 0$



L
L<0
L+ε
ε = -L > 0
L+(-L) = 0
L-ε → 2L
0
L
2L

Prop: Suppose $a_n \to L$ and $a_n \geq 0$ for all n. Then $L \geq 0$.

Pf: Suppose to the contrary that $L < 0$.

Pick $N \in \mathbb{N}$ so that if $n \geq N$, $\qquad a_n \geq 0$

$[\varepsilon > 0]$ $\qquad L < 0$

$$|a_n - L| < -L. \qquad a_n - L$$

In particular, $|a_N - L| < -L$ and

$$2L < a_N < 0.$$

$$\left[\begin{array}{ll} |a-b|<c & b-c<a<b+c \\ |a|<c & -c<a<c \end{array}\right]$$

Thus $a_N < 0$. But $a_N \geq 0$, a contradiction $\square$

Cor: If $a_n \to a$, $b_n \to b$ and if $a_n \geq b_n$ for all n then $a \geq b$.

Sketch: $a_n - b_n \geq 0$

$a_n - b_n \to a - b$

$\Rightarrow a - b \geq 0 \Rightarrow a \geq b.$

Cor: If $a_n \geq c$ for all n and if $a_n \to a$ then $a \geq c.$

Exercise.

Dealing with sequence convergence

without knowing what the limit is.

$x_1 = 3$

$x_2 = 3.1$

$x_3 = 3.14$

$x_4 = \underline{3.141}$

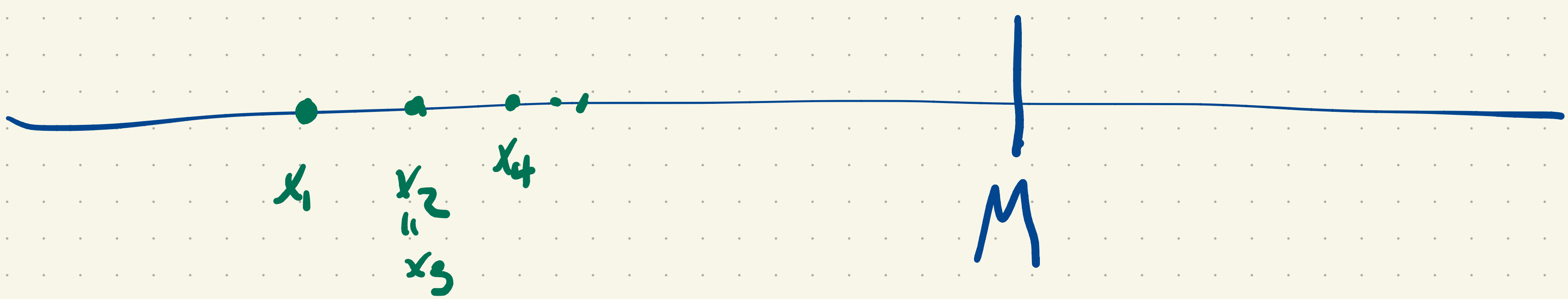
$\vdots$

Montone sequences.

Def: A sequence (a_n) is monotone increasing if $a_{n+1} \geq a_n$ for all n.

$a_n = 5 \ \forall n$ is monotone increasing.

$a_n = n$ Monotone increasing. Does not converge.



Prop: Suppose (a_n) is a monotone increasing sequence and there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all n. (We call M an upper bound for the sequence)

Then $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

$\rightarrow$ [AOC: i) $A \neq \emptyset$
ii) A is bounded above]

Pf: Let $A = \{\underline{a_n} : n \in \mathbb{N}\}$. Observe

$A \neq \emptyset$ since $a_1 \in A$. The set

A admits M as an upper bound.

Hence, by the AoC, A has a supremum, s.

We claim $a_n \to s$.

Let $\varepsilon > 0$. [Job: find an N that works:
$|a_n - s| < \varepsilon$ for $n \geq N$].

Since $s - \varepsilon < s$, $s - \varepsilon$ is not an upper bound for A. Hence there exists $a_N \in A$ such that $s - \varepsilon < a_N$.

Now if $n \geq N$ then $a_{N+2} \geq a_{N+1} \geq a_N$

$$s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon.$$

That is, if $n \geq N$, $|a_n - s| < \varepsilon$. $\square$