$a_n \to a \qquad b_n \to b$

$a_n b_n \to ab$

$$|ab - a_n b_n| = |ab - a_n b + a_n b - a_n b_n|$$

$$\leq |ab - a_n b| + |a_n b - a_n b_n|$$

$$= \underbrace{|a - a_n|}\,|b| + \underbrace{|a_n|\,|b - b_n|}_{\infty \cdot 0}$$

$< \frac{\varepsilon}{2|b|}$

$< \frac{\varepsilon}{2}$

$< \frac{\varepsilon}{2}$

$|b - b_n|\,|a_n| < \frac{\varepsilon}{2}$

$|b - b_n| < \varepsilon / 2|a_n|$

Def: A sequence (a_n) is __bounded__ if there exists

$M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n.

$\left(\frac{1}{n}\right)$ $\quad \left|\frac{1}{n}\right| \leq 1 \quad \forall n$

$n \geq 1 \quad \forall n \in \mathbb{N}$

Lemma: $1 \geq 1/n \quad \forall n \in \mathbb{N}$

$1 \geq 1/n \geq 0 \geq -1$

$1 \geq \frac{1}{n} \geq -1 \Rightarrow \left|\frac{1}{n}\right| \leq 1$

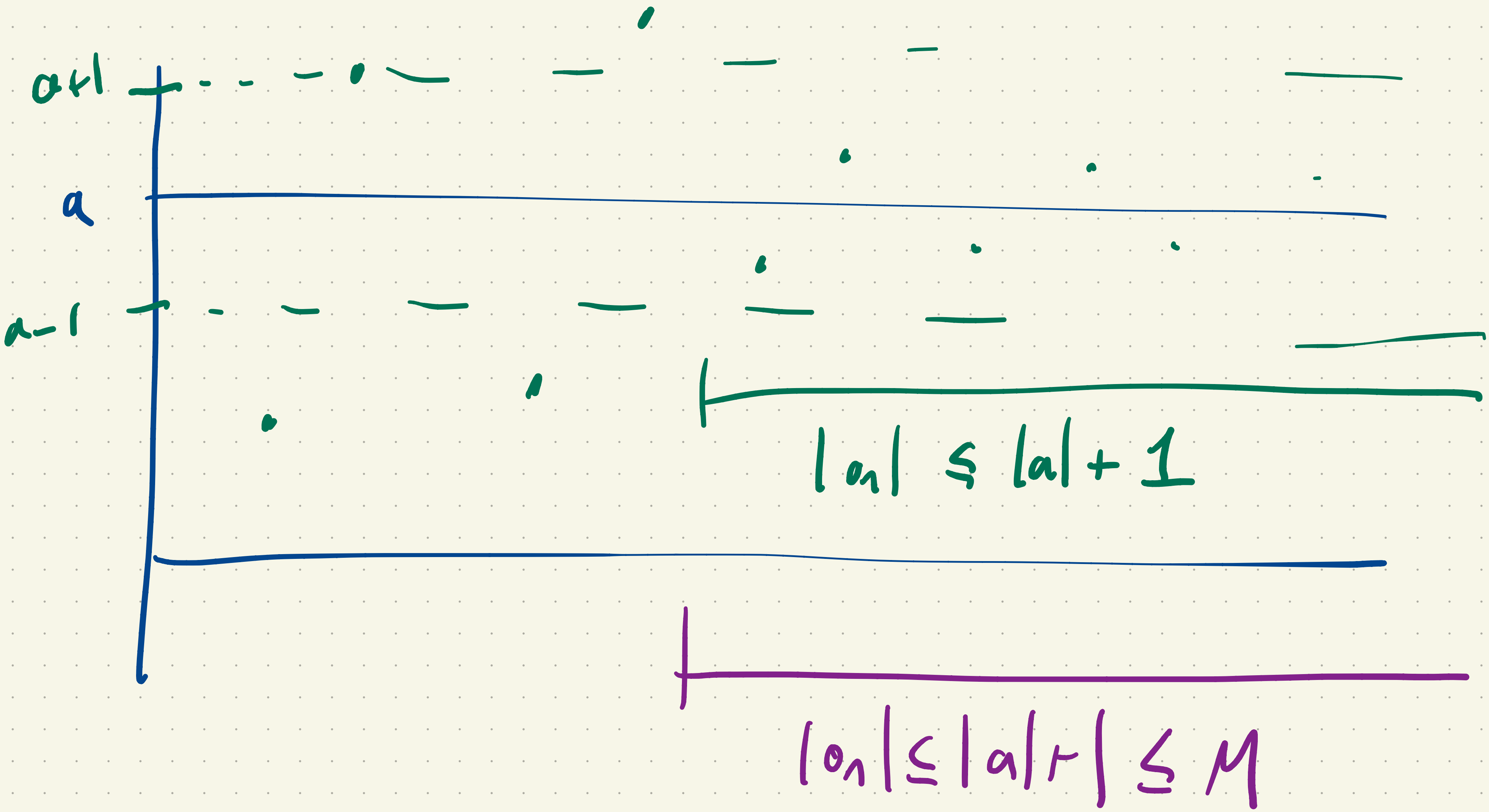
Unbonded: $a_n = n$

for all $M \in \mathbb{R}$ there exists $n \in \mathbb{N}$

so $|a_n| > M$. $M > 0$

$n > M$

Lemma: Suppose $(a_n) \to a$, Then the sequence is bounded.


$a+1$
a
$a-1$
$|a_n| \leq |a|+1$
$|a_n| \leq |a|+1 \leq M$

Pf: Since $a_n \to a$ there exists $N \in \mathbb{N}$ so that if $n \geq N$, $|a - a_n| < 1$.
is our ε

But then if $n \geq N$,

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a|$$
$$< 1 + |a|.$$

Let $M = \max(|a_1|, |a_2|, \dots, |a_N|, |a| + 1)$.

I claim that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Indeed, if $n \leq N$ then $|a_n| \leq M$ obviously.

Otherwise $n \geq N$ and we have argued

$$|a_n| \leq |a|+1 \leq M.$$

$\square$

Moral: Convergent sequences are bounded.

Prop: If $a_n \to a$ and $b_n \to b$ then

[case $a \neq 0$] $a_n b_n \to ab$.

Pf: Let $\varepsilon > 0$. [Job: show there is an N that works].

[N that works: if $n \geq N$

then $|ab - a_n b_n| < \varepsilon$]

Since (b_n) converges, it is bounded and there exists $M > 0$ so that $|b_n| \leq M$ for all n.

Pick $N_1 \in \mathbb{N}$ so that if $n \geq N_1$

$$|a_n - a| < \frac{\varepsilon}{2M}.$$

Pick $N_2 \in \mathbb{N}$ so if $n \geq N_2$

$$|b_n - b| < \frac{\varepsilon}{2|a|}. \quad (\text{This uses } a \neq 0).$$

But then if $n \geq \max(N_1, N_2)$ then

$$|ab - a_n b_n| = |ab - ab_n + ab_n - a_n b_n|$$

$$\leq |ab - ab_n| + |ab_n - a_n b_n|$$

$$= |a|\,|b - b_n| + |a - a_n|\,|b_n|$$

$$\leq |a|\,|b - b_n| + |a - a_n| M$$

$$< |a| \frac{\varepsilon}{2|a|} + \frac{\varepsilon}{2M} \cdot M$$

$$= \varepsilon.$$

$\square$

$$|a - a_n| M < \frac{\varepsilon}{2}$$

$$|a - a_n| < \frac{\varepsilon}{2M}$$

$a_n \to a$

$$\frac{1}{b_n} \longrightarrow \frac{1}{b} \qquad b_n \to b$$

$$\left| \frac{1}{b} - \frac{1}{b_n} \right| = \left| \frac{b_n - b}{b\, b_n} \right|$$

$$\leq \underbrace{|b_n - b|}_{\text{small}} \cdot \frac{1}{|b|\,|b_n|}$$

$$\frac{1}{|b|} \quad \frac{1}{|b_n|}$$

Lemma: Suppose $b_n \neq 0$ $\forall n$ and

$$\underline{b_n \to b} \neq 0.$$

Then there exists $M > 0$ so $\left|\frac{1}{b_n}\right| \leq M$

for all $n \in \mathbb{N}$.

Pf: Pick $N \in \mathbb{N}$ so if $n \geq N$

$$|b - b_n| < \frac{|b|}{2}.$$

Then if $n \geq N$,

$$|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n|$$

$$< \frac{|b|}{2} + |b_n|.$$

So if $n \geq N$, $|b_n| > |b|/2$ and

$$\frac{2}{|b|} > \frac{1}{|b_n|}.$$

Let $M = \max\left(\frac{1}{|b_1|}, \frac{1}{|b_2|}, \ldots, \frac{1}{|b_N|}, \frac{2}{|b|}\right).$

Then for all $n \in \mathbb{N}$, $\dfrac{1}{|b_n|} \leq M$. $\square$