Def: We say a sequence (x_k) converges to a limit $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n \geq N$ then $|L - x_n| < \varepsilon$.

If (x_k) converges to L we write

$$\lim_{k \to \infty} x_k = L \quad \text{or simply} \quad x_k \to L.$$

A sequence diverges if it does not converge to any real number.

A_n at most countable

$\emptyset$, finite, countably inf

A is at most countable $\Leftrightarrow$ there is a surjection $\Rightarrow$ at most countable

$f: \mathbb{N} \to A$

$\mathbb{N} \to S_n$

$1 \to 1$

$2 \to 2$

$\vdots$

$n \to n$

$j \to 1$

$A_1 \quad A_2 \quad A_3$

A_n at most contable.

↳ There exists a surjection $f_n : \mathbb{N} \to A_n$

$$\bigcup_{n=1}^{\infty} A_n$$

$$f : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$$

surjection

$\mathbb{N} \times \mathbb{N}$

surjection

$$g : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$$

$$\begin{tikzcd} \mathbb{N} \arrow[r] \arrow[r, bend left] & \mathbb{N} \times \mathbb{N} \arrow[r] \arrow[r, bend right] & \bigcup_{n=1}^{\infty} A_n \end{tikzcd}$$

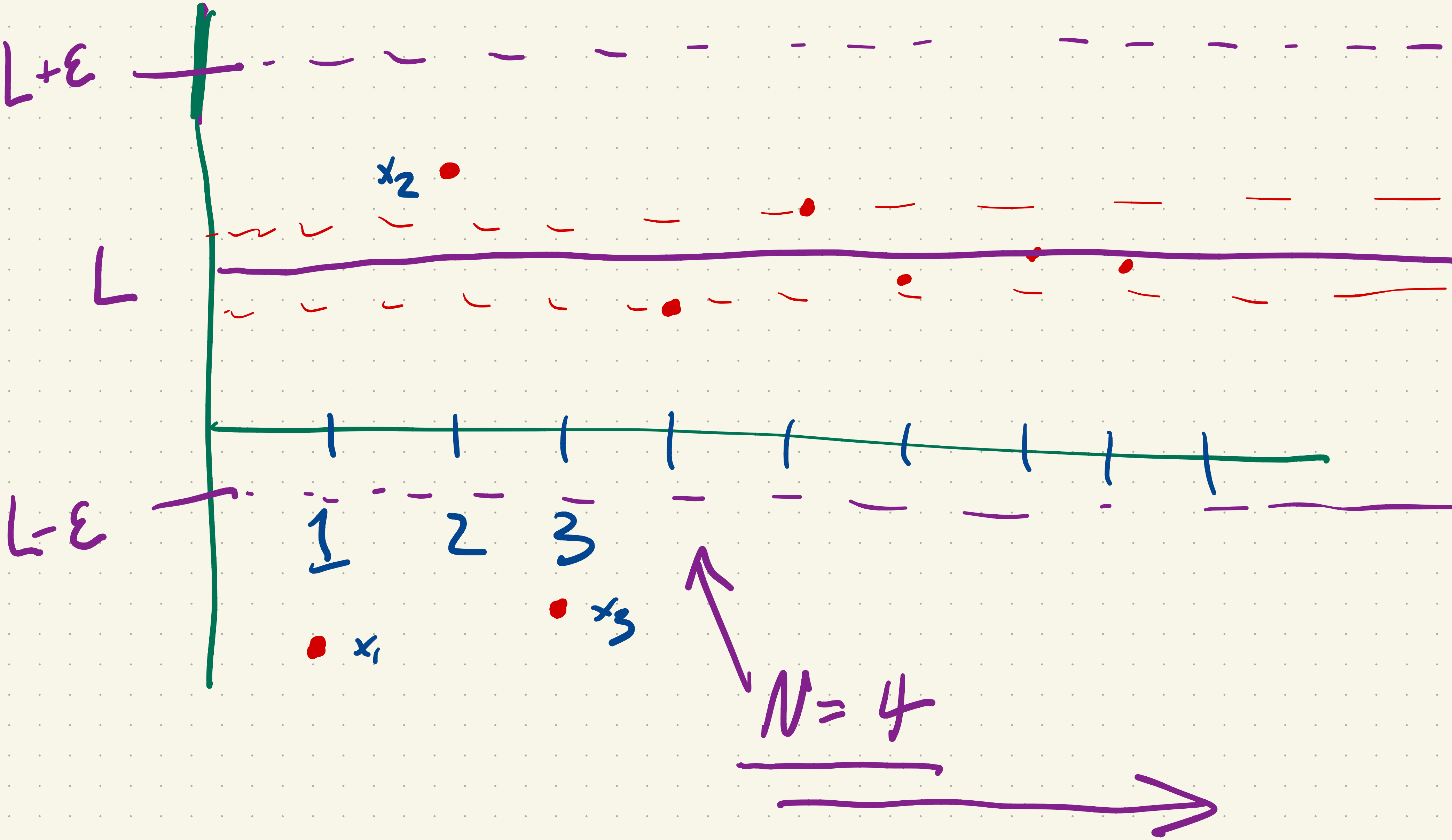
$$\mathbb{N} \xrightarrow{f_1} A_1$$

$$\mathbb{N} \xrightarrow{f_2} A_2$$

$$\mathbb{N} \xrightarrow{f_3} A_3$$

$$\mathbb{N} \xrightarrow{f_4} A_4$$

$$\vdots$$


L+ε
L
L-ε
x2
x1
x3
1
2
3
N=4

E.g. $x_n = \frac{1}{n}$

Claim $\lim_{n \to \infty} \frac{1}{n} = 0$.

Pf: Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then if $n \geq N$,

$$|0 - x_n| = \left|0 - \frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$\square$

$x_n = \frac{1}{n^2}$ $\qquad \lim_{n\to\infty} x_n = 0$ $\qquad \lim_{n\to\infty} \frac{1}{n} = 0$

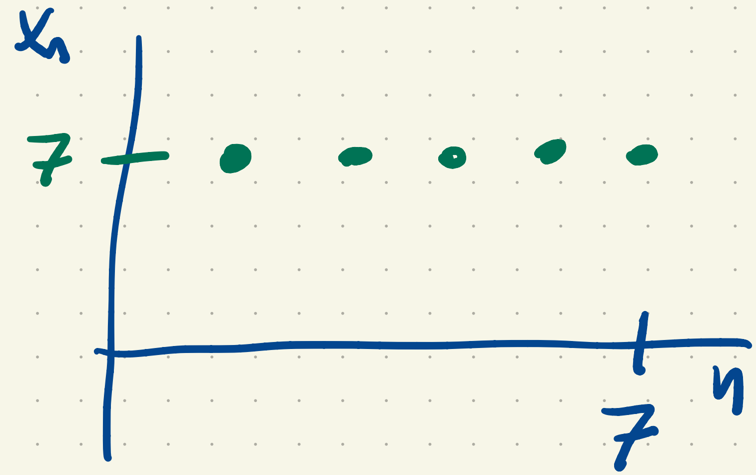
Claim $x_n \longrightarrow 0$

Pf: Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then if $n \geq N$

$$|0 - x_n| = \left|0 - \frac{1}{n^2}\right| = \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$\square$

$x_n = 7$ for all n



Claim: $\lim_{n \to \infty} x_n = 7$

Pf: Let $\varepsilon > 0$. Then if $n \geq \underline{1}$

$$|7 - x_n| = |7 - 7| = |0| = 0 < \varepsilon.$$

$\square$

Next HW: $(-1)^n$ does not converge.

For all $L \in \mathbb{R}$, $(-1)^n$ does not converge to L.

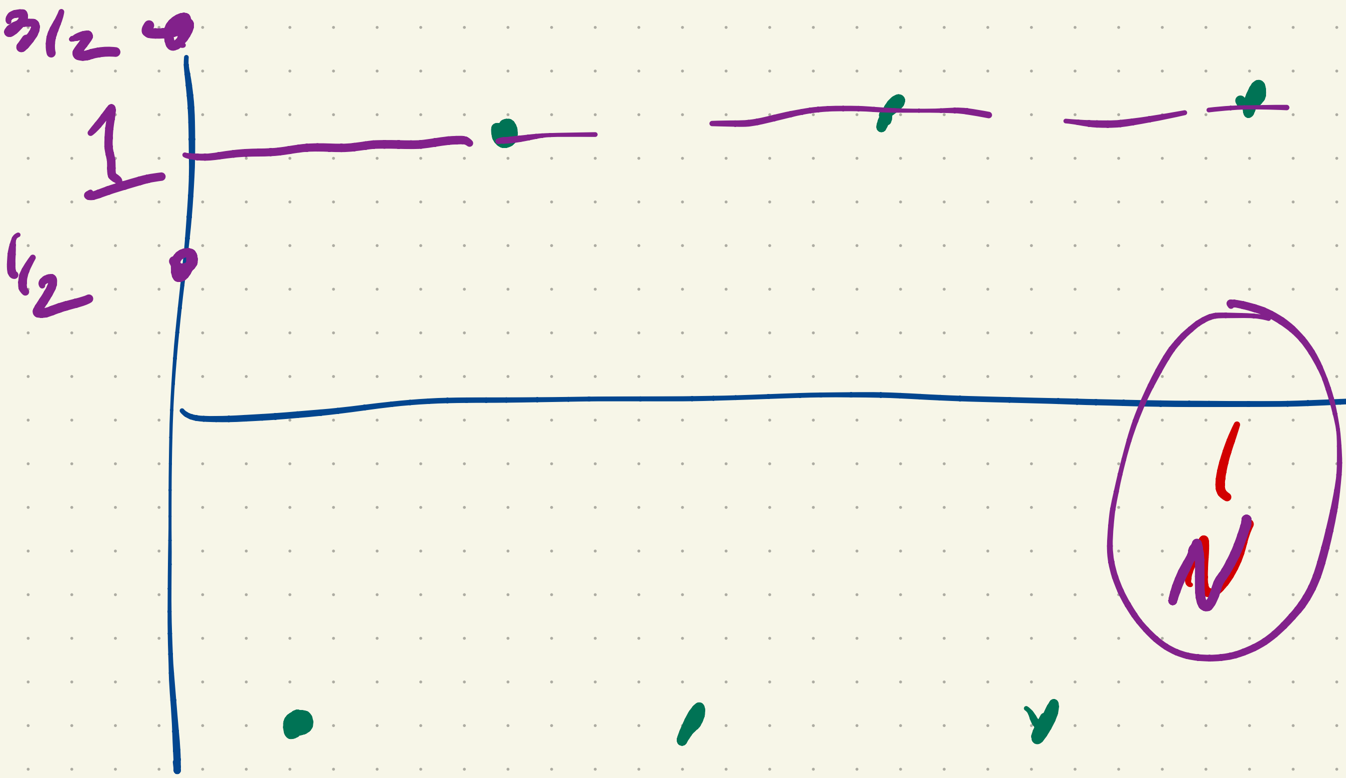
$(-1)^n$ converges to L:

For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so if $n \geq N$,

$$|L - (-1)^n| < \varepsilon.$$

$(-1)^n$ does not converge to L:

There exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$, $|L - (-1)^n| \geq \varepsilon$.



$n = N$ or $N+1$

n $\quad n = N?$

$x_N = 1$
or
$x_N = -1$

On: HW: Limits are unique.

$$x_n \to a,\quad x_n \to b \implies a = b$$

New sequences from old:

(a_n) $\quad a_n \to a$

(b_n) $\quad b_n \to b$

Facts:

1) $a_n + b_n \longrightarrow a+b$

2) $a_n \cdot b_n \longrightarrow ab$

3) $1/b_n \longrightarrow 1/b$

So long as $b \neq 0$

(and $b_n \neq 0 \ \forall n$)

$a_n \to a \qquad b_n \to b$

Let $\varepsilon > 0$.

$$|(a+b) - (a_n+b_n)| = |(a-a_n) + (b-b_n)|$$

$$\leq \underbrace{|a-a_n|} + \underbrace{|b-b_n|}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \to \varepsilon$$

$a_n \to a$, $b_n \to b$

Pf: Let $\varepsilon > 0$. Pick $N_1 \in \mathbb{N}$ so if $n \geq N_1$ then $|a - a_n| < \varepsilon/2$.

Pick $N_2 \in \mathbb{N}$ so if $n \geq N_2$ then $|b - b_n| < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$. Then if $n \geq N$, (and therefore $n \geq N_1$ and $n \geq N_2$)

$$
\begin{aligned}
|(a+b) - (a_n + b_n)| &= |(a - a_n) + (b - b_n)| \\
&\leq |a - a_n| + |b - b_n| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon. \qquad \square
\end{aligned}
$$

Prop: Suppose $a_n \to a$ and $b_n \to b$.

Then $a_n + b_n \to a + b$.

Show: $\forall\ \varepsilon > 0$ there exists

$N \in \mathbb{N}$ so if $n \geq N$,

$$|(a+b) - (a_n + b_n)| < \varepsilon,$$

$$f : \mathbb{N} \times \mathbb{N} \to \bigcup A_k$$

$$f_j : \mathbb{N} \to A_j \; , \quad \text{surjection}$$

$$f(i,j) = f_j(i)$$

Need to show f is a surjection:

Let $a \in \bigcup A_k$. ~~Then there exists~~

~~$i, j \in \mathbb{N}$ such that $f(i,j) = a$.~~

Then there exist $j \in \mathbb{N}$ such that $a \in A_j$. Since $f_j : \mathbb{N} \to A_j$ is a surjection there exists $i \in \mathbb{N}$ such that $f_j(i) = a$. Hence $f(i, j) = a$ and f is a surjection.

$\square$