Thm $\mathbb{R}$ is uncountable.

[$\mathbb{R}$ is infinite, but not countably infinite]

→ $A \subseteq B$, A is infinite $\Rightarrow$ B is infinite

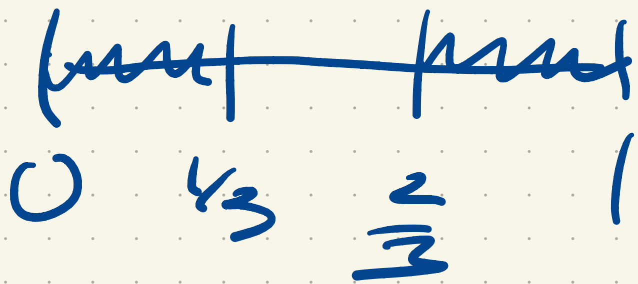
→ There does not exist

$f: \mathbb{N} \to \mathbb{R}$ that is a bijection

Pf: Suppose to the contrary that $f: \mathbb{N} \to \mathbb{R}$ is a bijection. For brevity let us write $x_k = f(k)$.

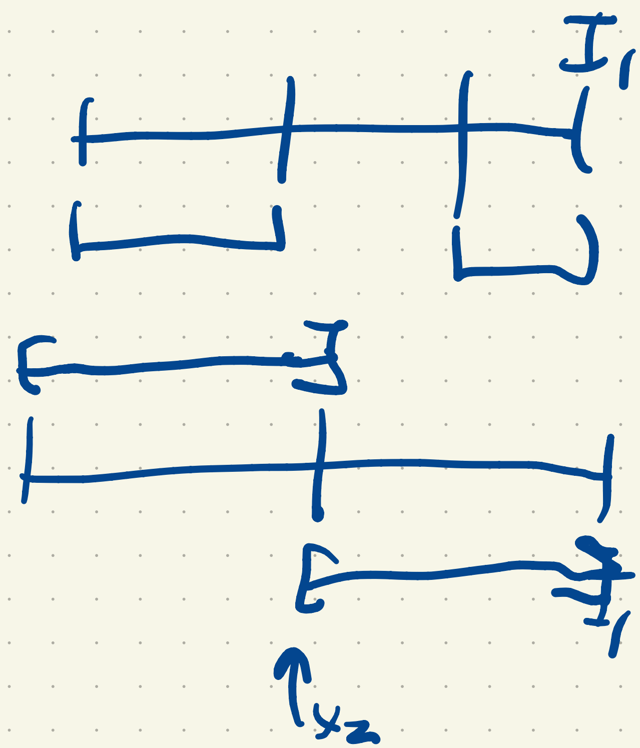
Define $I_0 = [0, 1]$.

One of the the two intervals $[0, 1/3]$ and $[2/3, 1]$ does not contain x_1.



Pick I_1 to be one of these intervals.

Similarly we can find a closed subinterval of I_1, I_2, that does not contain x_2



Continuing this process we can find nested closed intervals I_k such that $x_k \notin I_k$ for all $k \in \mathbb{N}$. By the Nested Interval Property there exists $x \in \bigcap_{k=1}^{\infty} I_k$. We know that $x = x_n$ for some n. By construction $x_n \notin I_n$ and hence $\underset{\substack{\| \\ x}}{x_n} \notin \bigcap_{k=1}^{\infty} I_k$. That is, $x \notin \bigcap_{k=1}^{\infty} I_k$.

This is a contradiction $\square$.

Alternative strategy

decimal expansions

$[0,1]$ is ~~not~~

uncountable.

$T = \{x \in \mathbb{R} : x^2 < 2\}$

$\hat{T} = \{q \in \mathbb{Q} : q^2 < 2\}$

$\hat{T}$ would have a sup. in $\mathbb{Q}$

$x \in [0,1]$ $\quad x = 0.d_1 d_2 d_3 d_4 \ldots\ldots$

$d_k \in \{0,1,2,\ldots,9\}$

$$x = \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10^4} + \cdots$$

$x = \frac{1}{2}$ $\quad x = 0.50000\ldots$

$0.49999\ldots$

$$0.9999\ldots = 1$$

$[0, 1]$

pick the all 9's if two expansions exist.

$x_1 = 0.d_{11}\, d_{12}\, d_{13} \ldots$

$x_2 = 0.d_{21}\, d_{22}\, d_{23} \ldots$

$x_3 = 0.d_{31}\, d_{32}\, d_{33} \ldots$

x_4

$\vdots$

$$d_k = \begin{cases} 5 & \text{if } d_{kk} = 7 \\ 7 & \text{otherwise.} \end{cases}$$

$$x = 0.\underbrace{d_1\, d_2\, d_3\, d_4 \ldots}_{5\text{'s } 7\text{'s}}$$

$x \neq x_k$

because $d_k \neq d_{kk}$

Sequences:

$x_1, x_2, x_3, x_4, \ldots$

Def: A (real-valued) sequence is a function $f: \mathbb{N} \to \mathbb{R}$.

$x_1 \quad x(1) \qquad x_k \quad x(k)$

sequences have a notion of first element and a notion of next.

$x_1 \to$ first element

x_{k+1} is next element after x_k.

$$\mathbb{Z}_{\geq k} = \{a \in \mathbb{Z} : a \geq k\}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$$

Notation

$$\{x_k\}_{k=1}^{\infty}$$

$$\{x_k\}$$

$$(x_k)$$

Convergence

What does it mean to say

$$x_k \longrightarrow z.$$

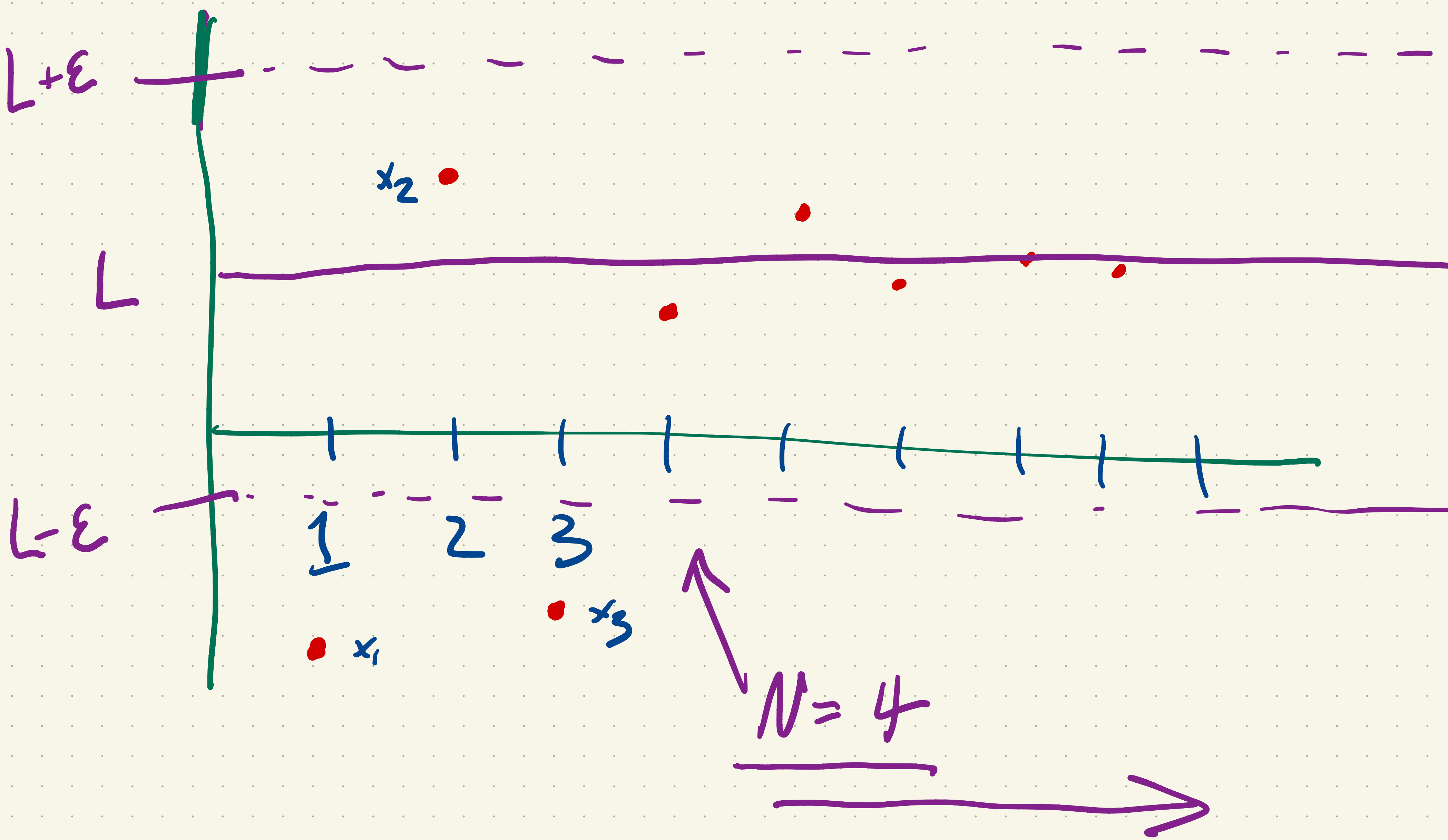
"x_k's get closer and closer to z"

Def: We say a sequence (x_k) converges to a limit $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n \geq N$ then $|L - x_n| < \varepsilon$.

If (x_k) converges to L we write

$$\lim_{k \to \infty} x_k = L \quad \text{or simply} \quad x_k \to L.$$

A sequence diverges if it does not converge to any real number.



L+ε
L
L-ε
x₂
x₁
x₃
1
2
3
N = 4

E.g. $x_n = \frac{1}{n}$

Claim $\lim_{n \to \infty} \frac{1}{n} = 0$.

Pf: Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that

$\frac{1}{N} < \varepsilon$. Then if $n \geq N$,

$$|0 - x_n| = \left|0 - \frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$\square$