A very little topology

Open intervals $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

Closed intervals $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$

Goal: generalize these concepts to other kinds ~~of sets~~

Def: Let $x \in \mathbb{R}$ and $\varepsilon > 0$. The ε-neighborhood of x is

$$V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon).$$

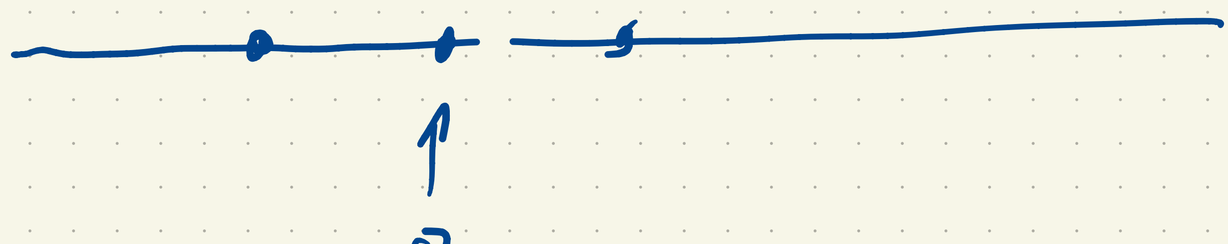
Exercise $V_\varepsilon(x) = \{ y \in \mathbb{R} : |x-y| < \varepsilon \}$

Def: A set $U \subseteq \mathbb{R}$ is <u>open</u> if for all $x \in U$ there exists $\varepsilon > 0$ such that

$$V_\varepsilon(x) \subseteq U.$$

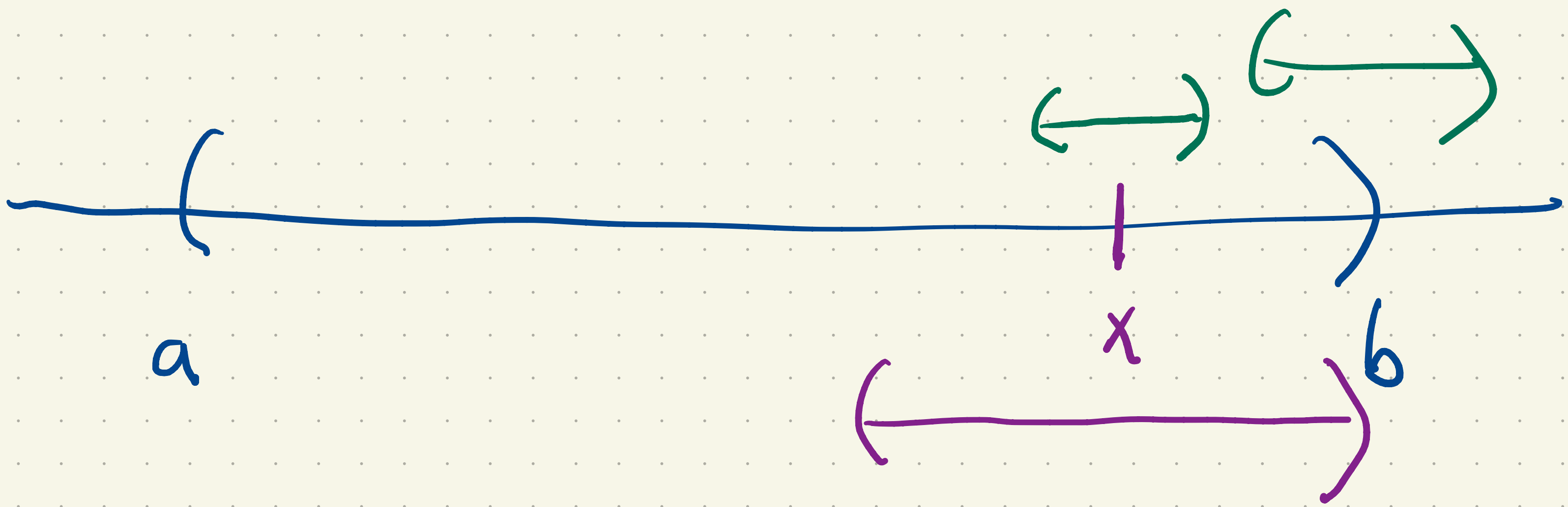
allowed to depend on x.

$\mathbb{Z} \subseteq \mathbb{R}$



$V_\varepsilon(0) \cap \mathbb{Z}$

$a = \quad (-1, 1)$



Lemma: Open intervals are open sets.

Proof: Consider an open interval (a,b).

Let $x \in (a,b)$. [Job: find ε ($V_\varepsilon(x) \subseteq (a,b)$)]

Let $\varepsilon = \min(b-x,\ x-a)$.

Suppose $y \in V_\varepsilon(x)$, so $\quad V_\varepsilon(x) \subseteq (a,b)$

$$x-\varepsilon < y < x+\varepsilon. \quad (1)$$

Observe $\varepsilon \leq b-x$ so $x+\varepsilon \leq b.^{(2)}$

Also $\varepsilon \leq x-a$ so

$$-\varepsilon \geq -x+a \quad \text{and}$$

$$x+\varepsilon \geq a. \quad (3)$$

Combining (1), (2) and (3) we find

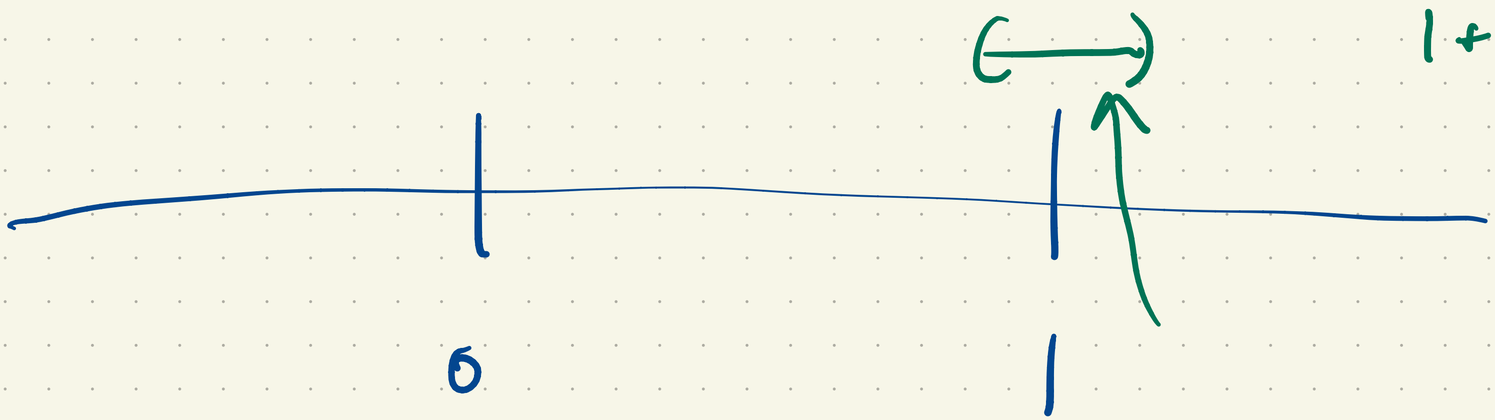
$a < y < b$ so $y \in (a,b)$.

$\square$

$[0,1]$

$(1-\varepsilon, 1+\varepsilon)$

$1 + \varepsilon/2 \notin [0,1]$



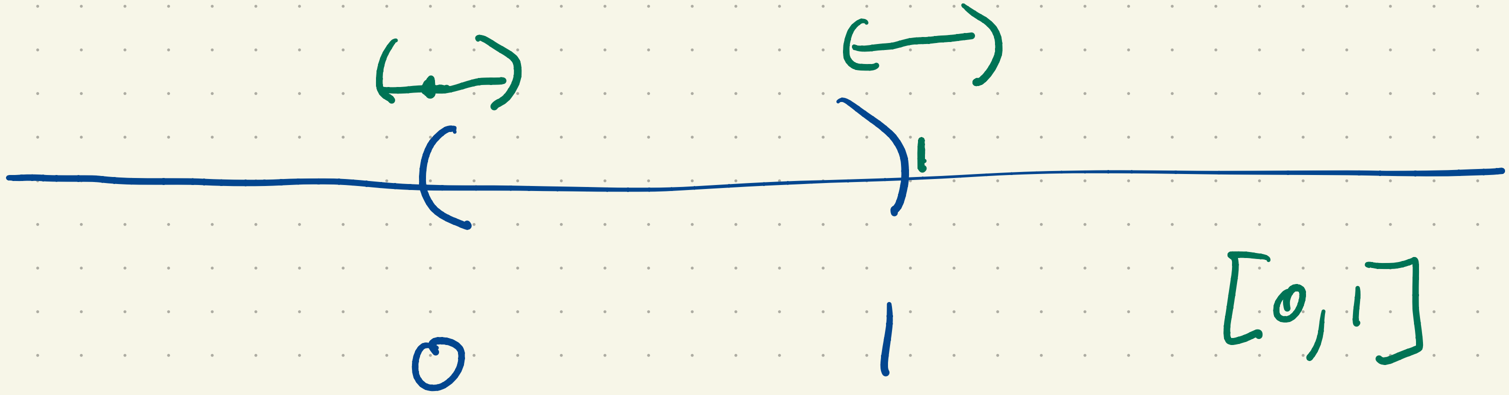
U is open: $\forall x \in U \ \exists \ \varepsilon > 0$ so $V_\varepsilon(x) \subseteq U$

U is not open: there exists $x \in U$ such that for all $\varepsilon > 0$ $\quad V_\varepsilon(x) \not\subseteq O$.

Def: Let $A \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ (possibly not in A) is a <u>limit point</u> of A if for every $\varepsilon > 0$

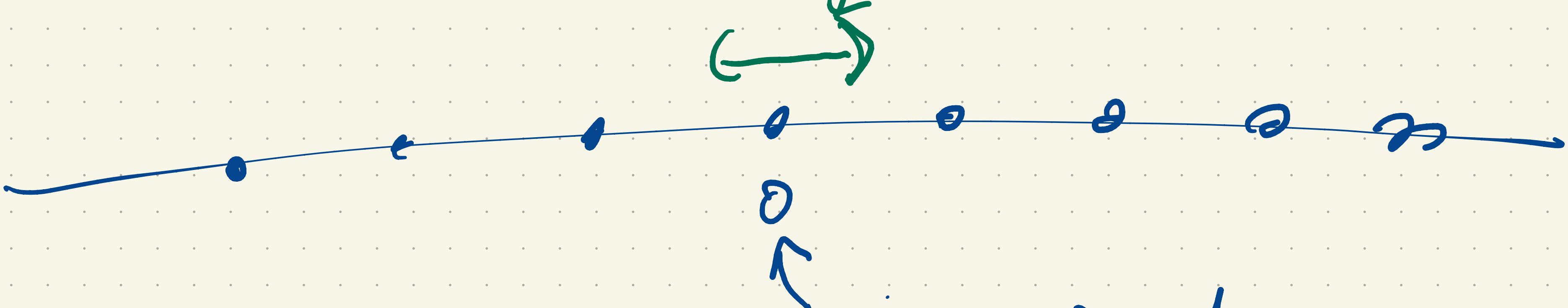
$$V_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset.$$

e.g. $A = (0,1)$. $\quad 0$ is a limit point



e.g. $A = \mathbb{Z}$

$U_{1/4}(0) = \left(-\frac{1}{4}, \frac{1}{4}\right)$

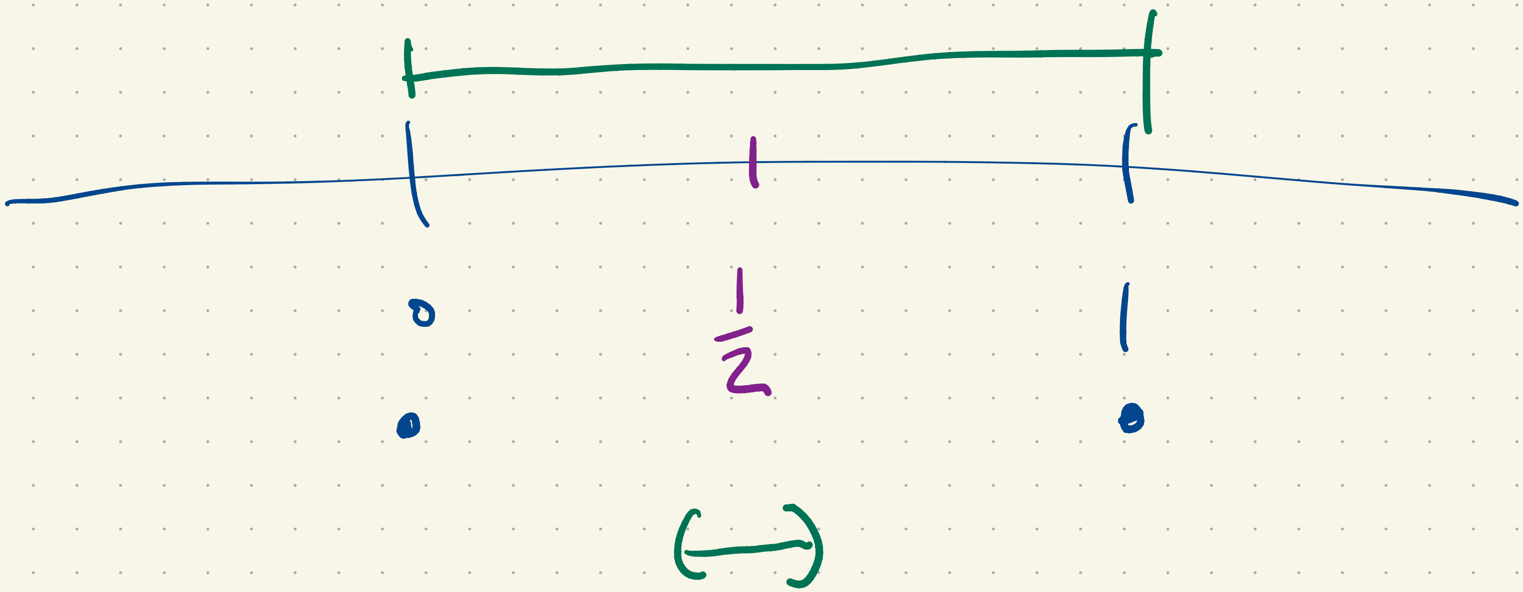


$0 \in A$

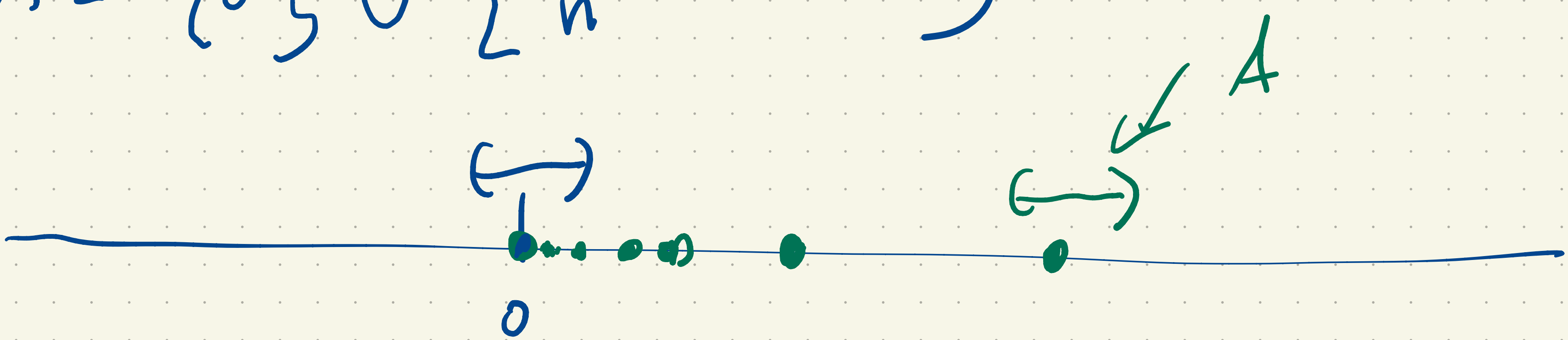
$U_{1/2}(0) \cap \mathbb{Z} = \{0\}$

$U_{\frac{1}{4}}(0) \cap \left(\mathbb{Z} \setminus \{0\}\right) = \emptyset$

e.g. $A = [0, 1]$

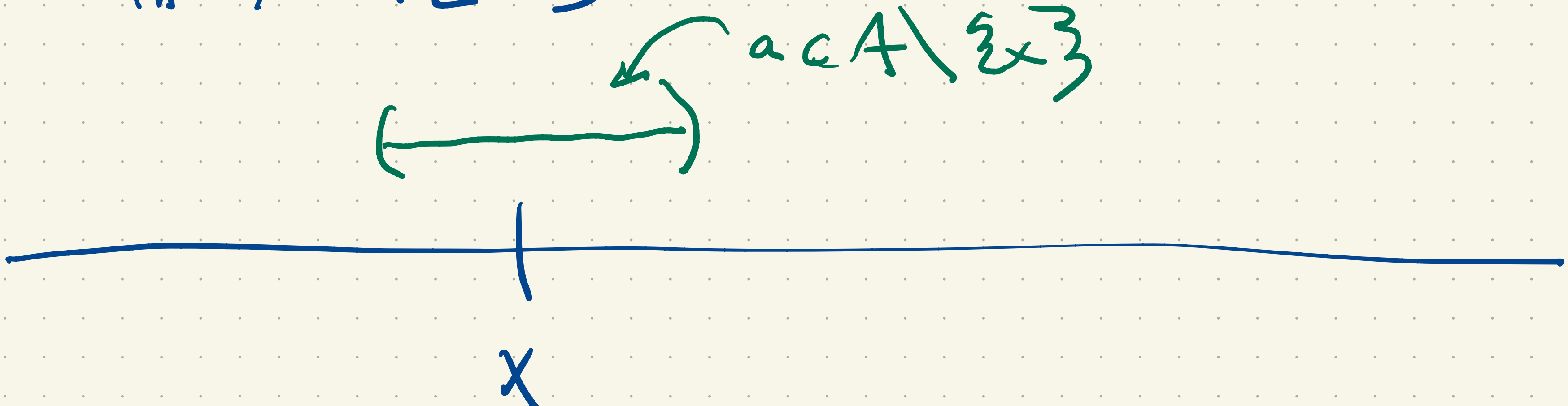


$$A = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

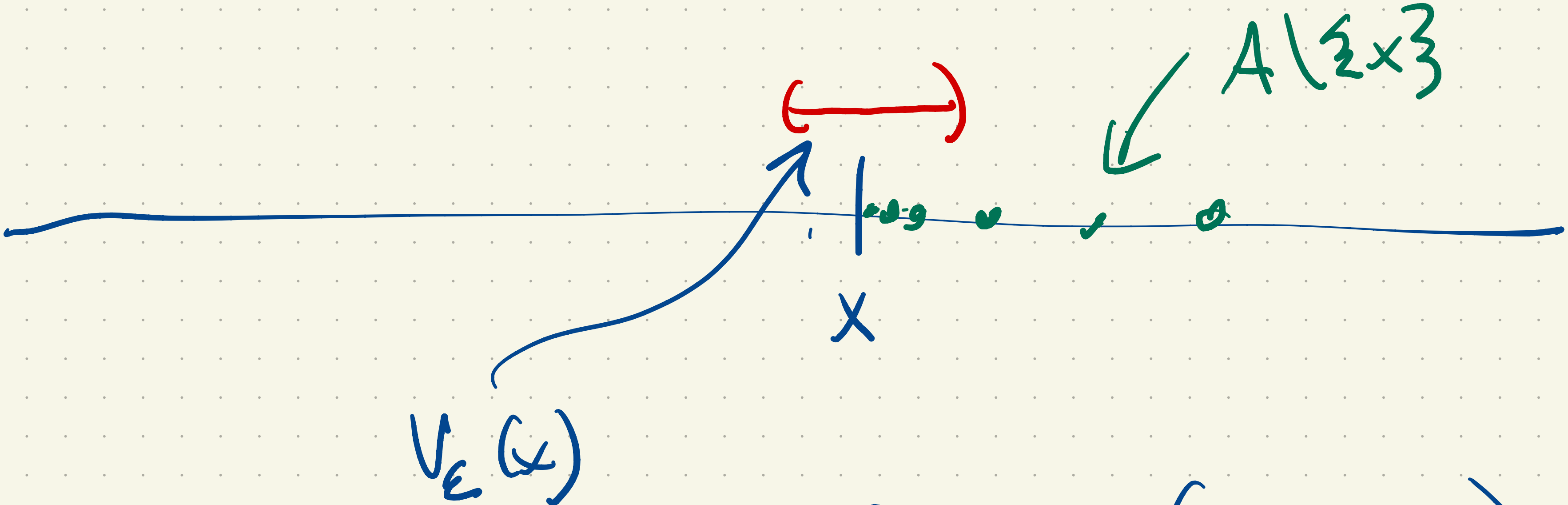


$\hat{A} = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

Thm: $x \in \mathbb{R}$ is a limit point of $A \subseteq \mathbb{R}$ if and only if there is a sequence in $A \setminus \{x\}$ that converges to x.



If x is a limit point of A, for each $n \in \mathbb{N}$ we can pick $a_n \in V_{\frac{1}{n}}(x) \cap A$



$$V_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset ?$$

$$a_n \to x$$

$$n \geq N, \quad |x - a_n| < \varepsilon$$

$$a_N \in A \setminus \{x\}$$

$$a_N \in V_\varepsilon(x)$$

Next class: closed sets

1) They contain their limit points

2) Their complements are open sets.