Last class: defined absolutely and cond. convergent series

$\sum_{k=1}^{\infty} a_k$ abs conv: $\sum_{k=1}^{\infty} |a_k|$ converges.

conditionally convergent: convergent, but not absolutely convergent

Does an absolutely convergent series converge?

$\sum_{k=1}^{\infty} a_k$ converges if it satisfies the CC for series.

If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent

$\Rightarrow$ convergent.

$\forall \varepsilon > 0$
$\exists N \in \mathbb{N}$ such that if
$n, m \geq N$
$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$$

Pf: Let $\varepsilon > 0$.

Since $\sum_{k=1}^{\infty} |a_k|$ converges

there exists N so if

$n > m \geq N$ then

$$\sum_{k=m+1}^{n} |a_k| < \varepsilon.$$

But then if $n > m \geq N$,

$$\left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| < \varepsilon.$$

$\square$

$$\left| \underline{a_{m+1} + a_{m+2} + \cdots + a_n} \right| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|$$

$$\sum_{k=1}^{\infty} a_k \qquad a_k = a_k^+ - a_k^-$$

If $a_k \geq 0$, $a_k^+ = a_k$, $a_k^- = 0$

If $a_k \leq 0$, $a_k^+ = 0$ $a_k^- = -a_k$

$$a_k^+, a_k^- \geq 0$$

$$|a_k| = a_k^+ + a_k^-$$

If $a_k < 0$, $|a_k| = -a_k$

$a_k^+ \leq |a_k|$ $\qquad$ $a_k^+ + a_k^- = |a_k|$

$a_k^- \leq |a_k|$ $\qquad$ $a_k^+ \leq a_k^+ + a_k^- = |a_k|$

Comparison test: $0 \leq a_k \leq b_k$ (Monotone conv. Thm).

If $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converge.

If $\sum_{k=1}^{\infty} a_k$ diverges $\Rightarrow$ $\sum_{k=1}^{\infty} b_k$ diverge.

If $\sum_{k=1}^{\infty} |a_k|$ conv then

$$\sum_{k=1}^{\infty} a_k^+ \quad \text{converges}$$

$$\sum_{k=1}^{\infty} a_k^- \quad \text{converges.}$$

$$\underbrace{\sum_{k=1}^{\infty} a_k^+ - a_k^-}_{\sum_{k=1}^{\infty} a_k \,=} = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

If $\sum a_k$ conv. and $\sum a_k^+$ conv

$\Rightarrow \sum a_k^-$ conv also.

$a_k = a_k^+ - a_k^-$

$\Rightarrow \sum_{k=1}^{\infty} a_k = \infty$

$a_k^- = a_k^+ - a_k$

$$\sum_{k=1}^{\infty} a_k^- = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k$$

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

$$\infty - \infty$$

For an absolutely convergent series, every rearrangement of the series converges to the same limit.

Def We say that $\sum_{k=1}^{\infty} b_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$ if

Suppose $n \geq N$. Observe that

$t_n - s_{\hat{N}}$ is a sum of distinct

$b_1 + b_2 + \cdots + b_n$

$b_1 + b_2 + \cdots + b_N + [\quad\hat{}\quad]$

a_k's where $k > \hat{N}$.

So $\left| t_n - s_{\hat{N}} \right| \leq \sum_{k=\hat{N}+1}^{M} |a_k|$ for some M.

Observe that $\sum_{k=\hat{N}+1}^{M} |a_k| < \frac{\varepsilon}{2}$.

$\hat{N} \geq N_2$

Hence $\forall\ n \geq N_1$ $\qquad \hat{N} \geq N_1$

$$|L - t_n| = |L - s_{\hat{N}} + s_{\hat{N}} - t_n|$$

$$\leq |L - s_{\hat{N}}| + |s_{\hat{N}} - t_n|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

$a_1 + a_2 + \cdots + a_{\hat{N}}$ $\qquad$ $n \geq N$ $\qquad$ t_n

$$b_1 + b_2 + b_3 + b_4 + b_5 + \cdots + b_N + \underbrace{b_{N+1} + \cdots + b_n}_{a\text{'s}}$$

a's $\qquad$ a_k

$k > \hat{N}$