Thm: [Alternating Series Test] $a_n = \frac{1}{n}$

Suppose (a_n) is a monotone decreasing sequence that converges to 0. Then $a_{n+1} \leq a_n$

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

(a_n)

$a_{n+1} \leq a_n$

$$\frac{1}{n+1} \leq \frac{1}{n}$$

$n \leq n+1$

$a_n = 1 \quad \forall n$

$a_n = 1 + \frac{1}{n}$

Let (s_k) be the sequence of partial sums of the series.

Pf: First, observe since (a_n) is decreasing

for all $j \in \mathbb{N}$

$$s_{2j+1} = s_{2j-1} - a_{2j} + a_{2j+1}$$

$$\leq s_{2j-1}$$

and hence (s_{2j-1}) is monotone decreasing.

Similarly for all $j \in \mathbb{N}$

$$s_{2j+2} = s_{2j} + a_{2j+1} - a_{2j+2}$$

$$\geq s_{2j}.$$

So (s_{2j}) is monotone increasing.

Now, if $j \in \mathbb{N}$,

$$s_1 \geq s_{2j+1} = s_{2j} + a_{2j+1} \geq s_{2j}.$$

So s_1 is an upper bound for (s_{2j}).

Similarly $j \in \mathbb{N}$

$$s_2 \leq s_{2j} = s_{2j+1} - a_{2j+1} \leq s_{2j+1}$$

$$s_2 \leq s_{2j} = s_{2j-1} - a_{2j} \leq s_{2j-1}$$

and hence s_2 is a lower bound for (s_{2j-1}).

Hence (s_{2j}) is increasing and bounded above, So it converges to a limit L.

Similarly, (s_{2j-1}) converges to a limit L'.

Taking the limit of the equation

$$s_{2j} = s_{2j-1} - a_{2j}$$

$a_j \to 0$

a_{2j}

$\uparrow$

subseq.

we conclude

$$L = L' - 0.$$

Since (s_k) is the shuffled sequence of (s_{2j-1}) and (s_{2j}), problem 2.3.5

implies $s_k \to L$.

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{vs} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$\frac{S}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} \cdots$$

$$= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 \cdots$$

$$\underbrace{S + \frac{S}{2}}_{\frac{3S}{2}} = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} - \cdots$$

$$\left[\frac{3S}{2} = S \text{ ???} \right]$$

$$\neq S \quad (!)$$

Def: A series $\sum_{k=1}^{\infty} a_k$ is <u>absolutely conv</u>.

if $\sum_{k=1}^{\infty} |a_k|$ is convergent. A convergent series that is not absolutely convergent is called <u>conditionally convergent</u>

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{n} \implies \text{conditionally conv.}$$

$\sum a_n$, convergent.

$$a_n^+ = \max(a_n, 0) = \begin{cases} a_n & , \text{if } a_n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_n^- = \max(-a_n, 0) = \begin{cases} -a_n & , \text{if } a_n \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$a_n^+ \geq 0$, $a_n^- \geq 0$

$$a_n = a_n^+ - a_n^-$$

$$|a_n| = a_n^+ + a_n^-$$

If $\sum |a_n|$ converges so does

$\sum a_n^+$ (comp. test.)

$\sum a_n^-$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - a_n^- = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$$

$$|a_n| = a_n^+ + a_n^-$$

Suppose $\sum a_n$ converges

and $\sum a_n^+$ converges.

$$a_n^+ - a_n^- = a_n$$

$$a_n^- = a_n^+ - a_n$$

$$\underline{\sum a_n^-} = \sum a_n^+ - a_n = \sum a_n^+ - \sum a_n$$

$\sum a_n^+ + a_n^-$ converges

if $\sum a_n$ converges and if

one of $\sum a_n^+$ or $\sum a_n^-$

converges, in which case

they both do.

Conditionally convergent:

↳ $\sum a_n$ converges but

$\sum a_n^+$ diverge

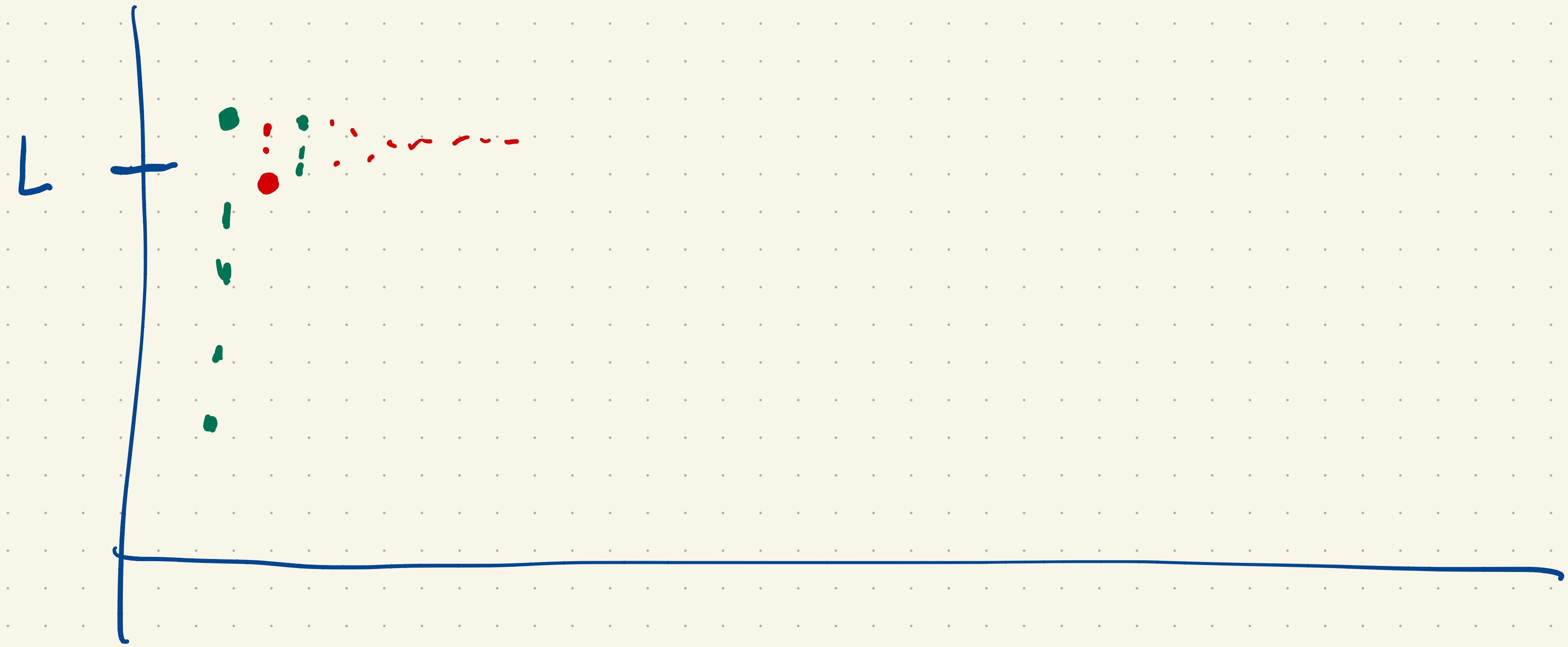
$\sum a_n^-$ diverge.

Absolutely convergent $\quad \sum a_n^+$ converges

$\sum a_n^-$ converge.

$$\sum a_n = \sum a_n^+ - \sum a_n^- \qquad \sum \frac{(-1)^{n+1}}{n}$$

$$\infty - \infty$$



$\sum a_n^+$

$\sum a_n^-$

$a_n \to 0$

$\sum a_n$

↳ convergent