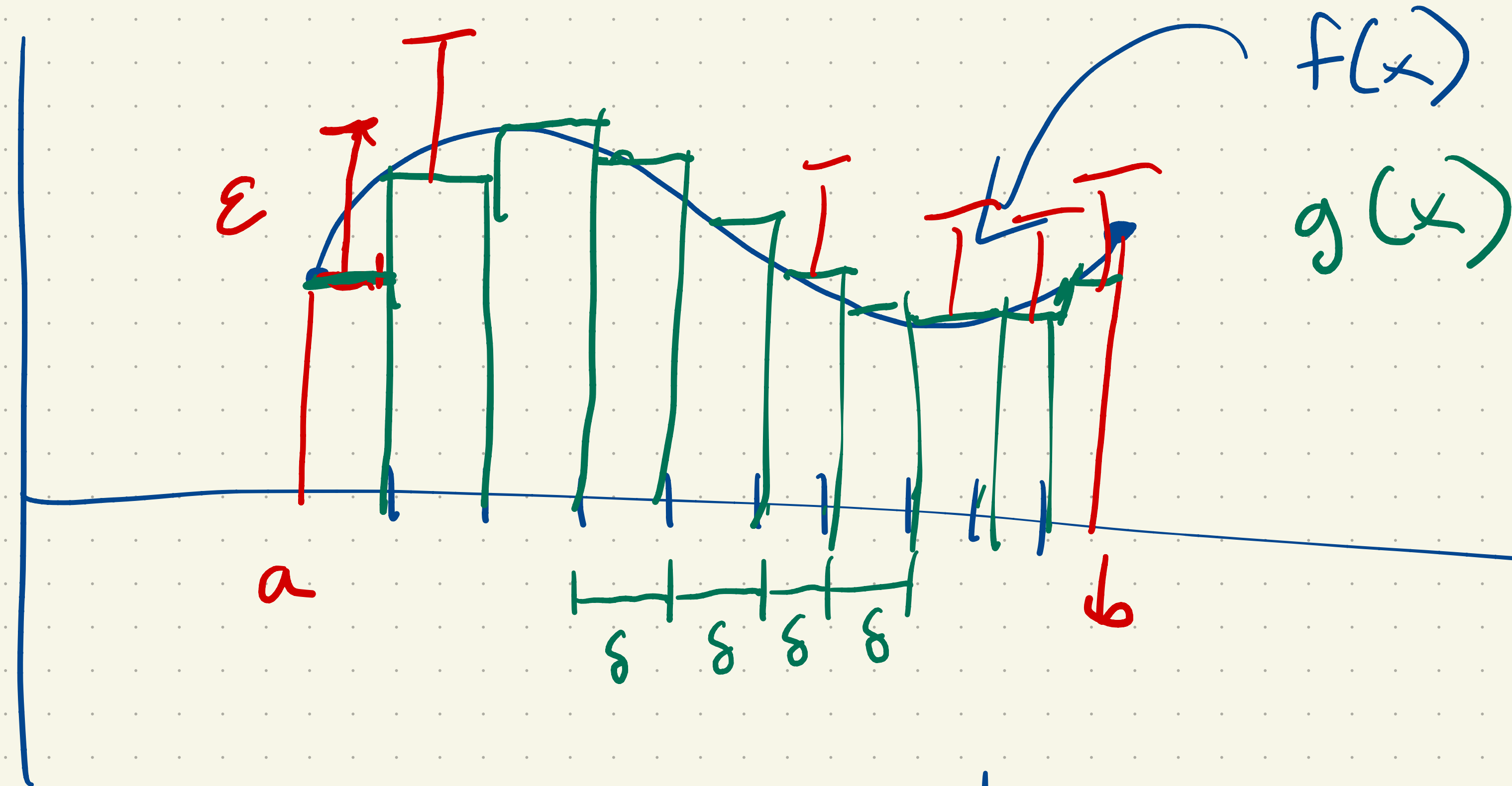


ε

$$|f(x) - g(x)| < \varepsilon$$

error in area: $\varepsilon \cdot (b-a)$

Continuity: $f: A \to \mathbb{R}$

$\forall a \in A$, $\forall \varepsilon > 0$, $\exists \delta > 0$ so if $x \in A$

$|a - x| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Uniform Continuity:

$\forall \varepsilon > 0$ there exists $\delta > 0$ so if

$a, x \in A$ and if $|a - x| < \delta$, then

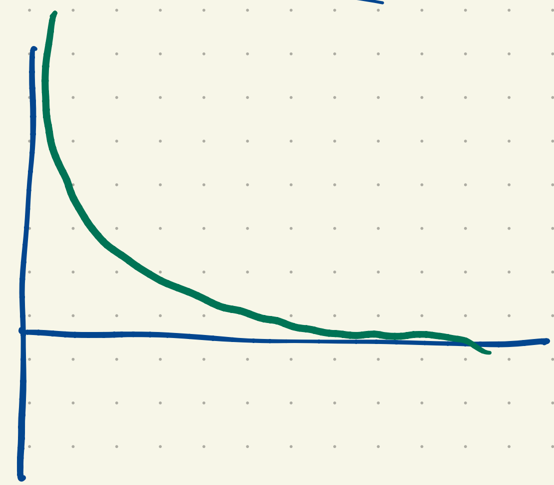
$|f(x) - f(a)| < \varepsilon$.

$x_n,\ z_n$ $\delta = 1/n$

Not uniformly continuous:

There there exist ε_0 such that for all $\delta > 0$ there exist $a, x \in A$ such that $|a - x| < \delta$ but $|f(x) - f(a)| \geq \varepsilon_0$.

$f(x) = \frac{1}{x}$ $\quad A = (0, \infty)$



$x_n = \frac{1}{n}$, $z_n = \frac{2}{n}$ $\quad n = 1, 2, 3$

$|z_n - x_n| \leq \frac{1}{n}$ $\quad |f(x_n) - f(z_n)| = \left|n - \frac{n}{2}\right| = \frac{n}{2}$

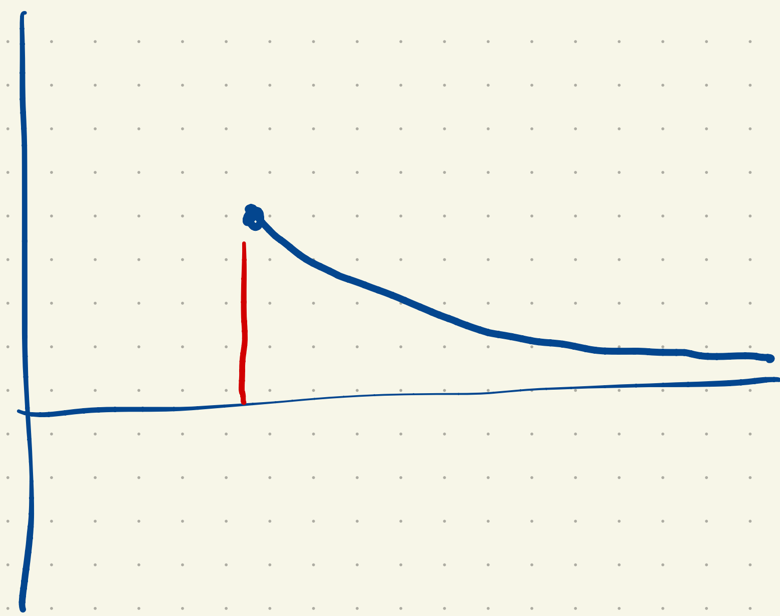
$\geq \frac{1}{2}$

$\varepsilon_0 = \frac{1}{2}$ $\delta > 0$

a, x $|a - x| < \delta$

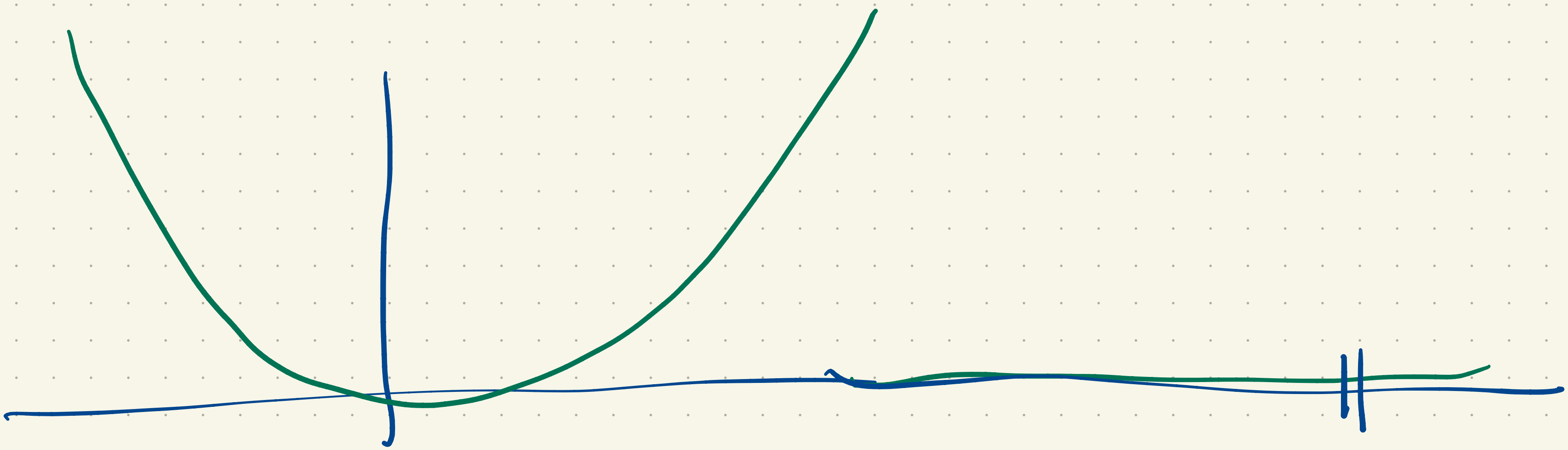
$\frac{1}{n} < \delta$ z_n x_n $|f(a) - f(x)| \geq \varepsilon_0$

$|z_n - x_n| < \delta$



$I \geq 2$

$f(x) = x^2$ on $\mathbb{R}$



$x_n = n + \frac{1}{n}$

$z_n = n$

$|x_n - z_n| \leq \frac{1}{n}$

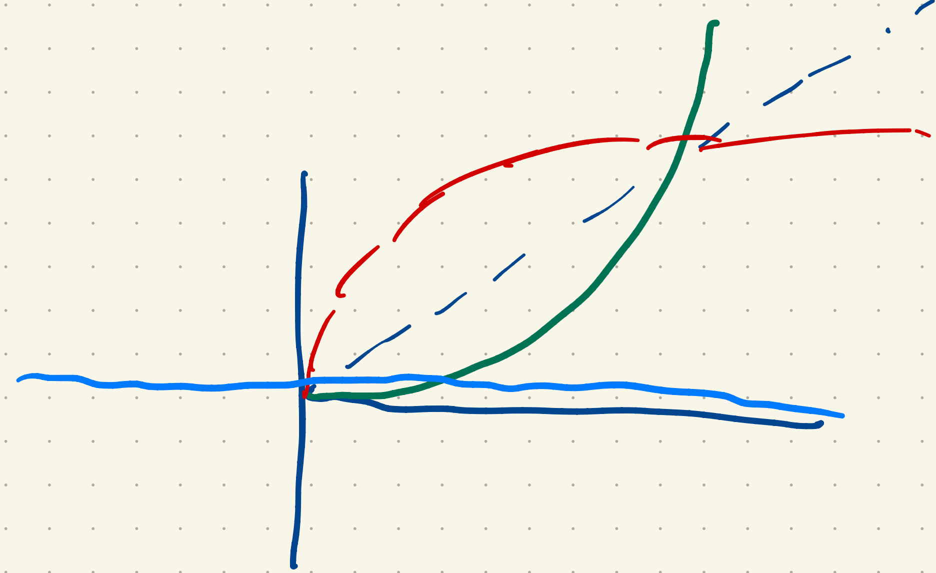
$f(x_n) = n^2 + 2n \cdot \frac{1}{n} + \frac{1}{n^2}$

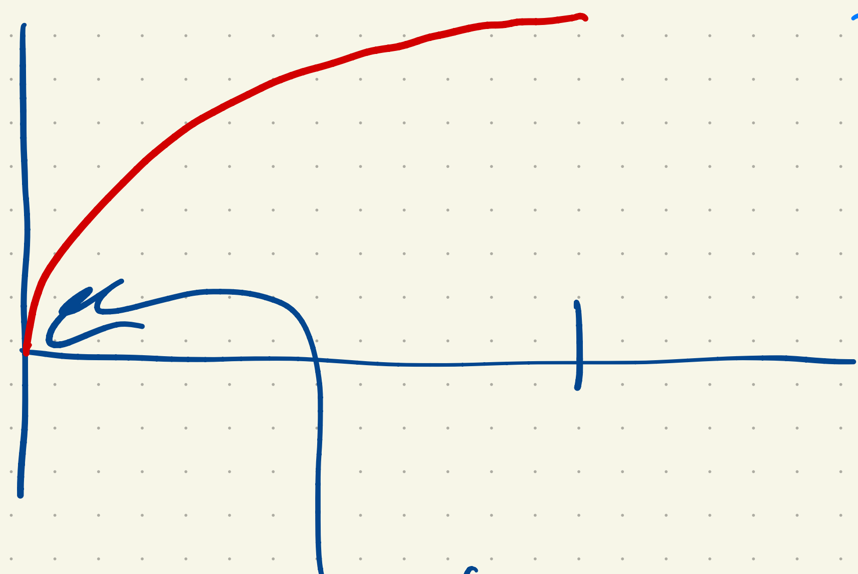
$= 2 + \frac{1}{n^2} + n^2$

$\underline{f(z_n) = n^2}$

$$|f(x_n) - f(z_n)| = 2 + \frac{1}{n^2} \geq 2$$

$f(x) = \sqrt{x}$ $[0,1]$





$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f'(0) =$$

x_n, z_n $\qquad x_n = \frac{1}{n} \quad z_n = \frac{2}{n} \quad |x_n - z_n| \leq \frac{1}{n}$

$$-(f(x_n) - f(z_n)) = -\left(\sqrt{1/n} - \sqrt{\frac{2}{n}}\right) \cdot \left(\frac{\sqrt{1/n} + \sqrt{2/n}}{\sqrt{1/n} + \sqrt{2/n}}\right)$$

$$= \frac{-\frac{1}{n} + \frac{2}{n}}{\sqrt{1/n} + \sqrt{2/n}}$$

$$= \frac{1}{\sqrt{n} + \sqrt{2n}} \leq \frac{1}{\sqrt{n}}$$

$$|f(x_n) - f(z_n)| \leq \frac{1}{\sqrt{n}}$$

Thm: Suppose $A \subseteq \mathbb{R}$ is compact and

$f: A \to \mathbb{R}$ is continuous.

Then f is <u>uniformly continuous</u>.

Pf: Suppose to the contrary that f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$ there exist $x_n, z_n \in A$ with $|x_n - z_n| < \frac{1}{n}$ but $|f(x_n) - f(z_n)| \geq \varepsilon_0$. Since A is compact there exists a subsequence x_{n_k} that converges to a limit $x \in A$.

Moreover, since $|z_{n_k} - x_{n_k}| \leq \frac{1}{n_k}$,

$$z_{n_k} - x_{n_k} \to 0$$

and $z_{n_k} \to x$ also.

$$z_{n_k} = \underset{\downarrow \atop x}{x_{n_k}} + \underset{\downarrow \atop 0}{\underbrace{(z_{n_k} - x_{n_k})}}$$

$$|\underbrace{z_n - x_n}_{w_n}| < \frac{1}{n}$$

$$0 \leq |w_n| < \frac{1}{n}$$

$$w_n \to 0$$

By continuity, $f(x_{n_k}) \to f(x)$

$$f(z_{n_k}) \to f(x).$$ Thus:

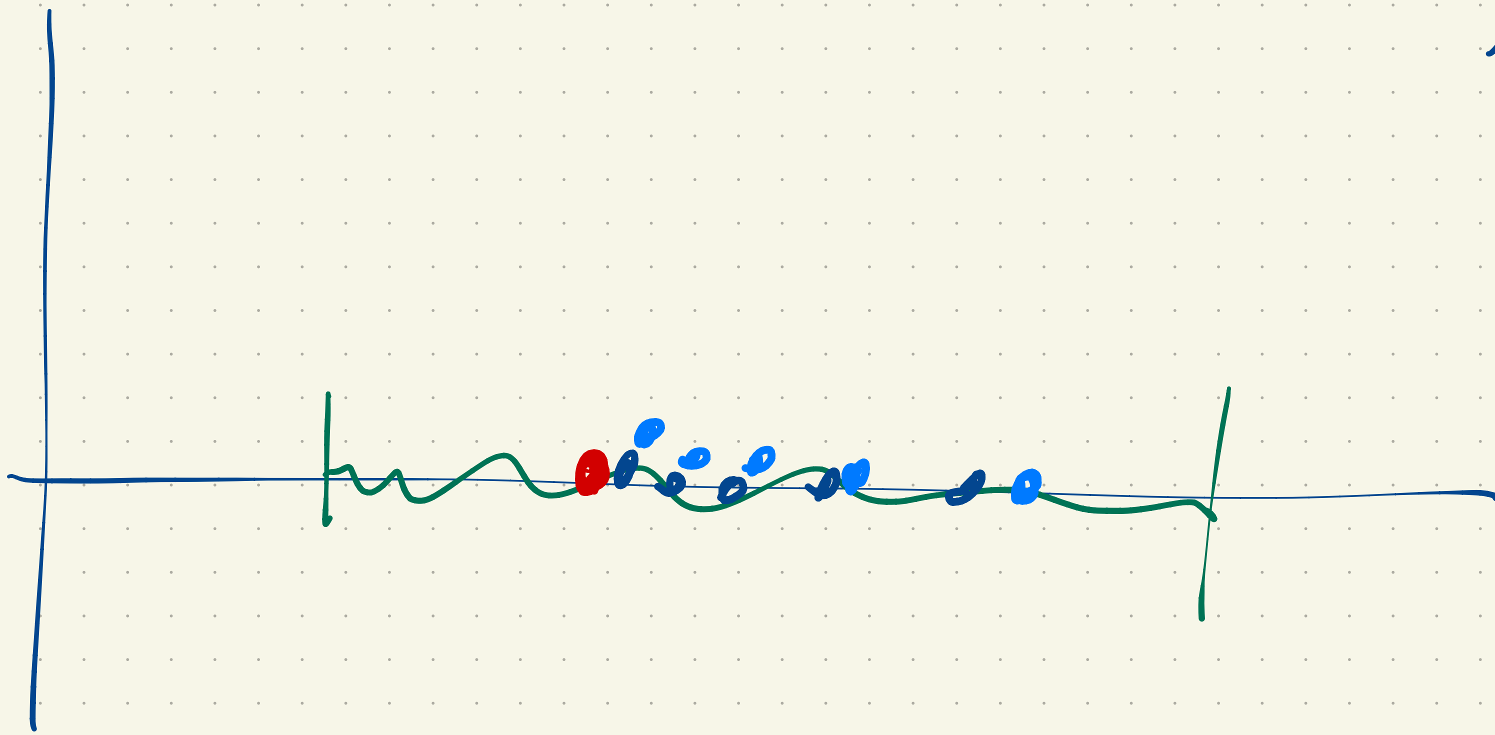
$$f(x_{n_k}) - f(z_{n_k}) \to 0$$

but $$\left| f(x_{n_k}) - f(z_{n_k}) \right| \geq \varepsilon_0 .$$

This contradicts the Order Limit Theorem.

$\square$

x_n, z_n



w_n

$w_n \to L$

$w_n \geq 5$

$\underbrace{\left| f(x_{n_k}) - f(z_{n_k}) \right|}_{w_k} \to 0$

$w_k \to 0$

$w_k \geq \varepsilon_0$

Thus $\exists$ K so if $k \geq K$,

$$\left| f(x_{n_k}) - f(z_{n_k}) \right| < \varepsilon_0 .$$

But this contradicts the fact that $\left| f(x_{n_k}) - f(z_{n_k}) \right| \geq \varepsilon_0$

$\forall k$.