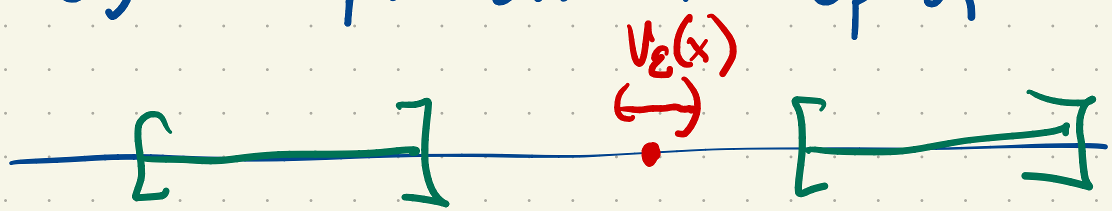
Compact set: closed + bounded.

$A \subseteq \mathbb{R}$. bounded $A \subseteq [-M, M]$ for some $M > 0$

closed set: a) contains its limit points

b) every convergent sequence in A converges to a limit in A.

c) complement is open



$A = (0, 1]$ is not closed

0 is a limit point of A that is not in A.

$x_n = \frac{1}{n}$ $\quad x_n \to 0 \notin A$

Bolzano-Weierstrass property

Every sequence in A has a subsequence that converges to a limit in A.

Compact sets have the B-W property.

A, compact

(x_n) in A.

A is compact $\Rightarrow$ bounded

$\Rightarrow A \subseteq [-M, M]$ for some M

$\Rightarrow |x_n| \leq M$ for all n.

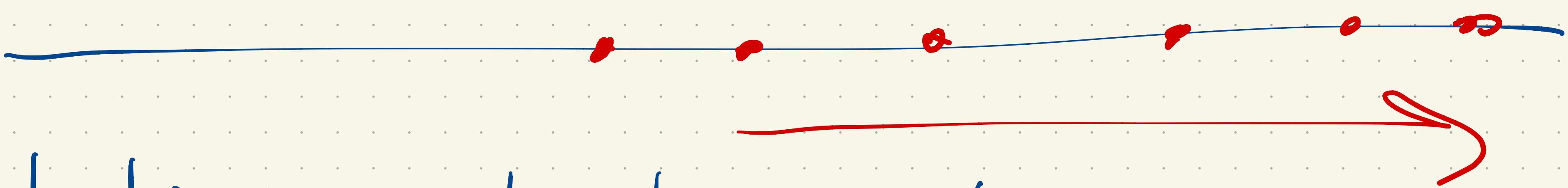
By BW there is a subsequence

$x_{n_k} \to L$ for some $L \in \mathbb{R}$.

Since A is closed, $L \in A$.

$\square$

If A is not bounded $\Rightarrow$ A does not have B-W property.



$|x_n| > n$ $\qquad$ $|x_{n_k}| > n_k \geq k$

If A is not closed $\Rightarrow$ A does not have B-W property,

c, limit point of A, $c \notin A$.

(x_n) $\quad x_n \in A \quad x_n \to c$

$x_{n_j} \to c \notin A$

$x_{n_j} \to a \in A$?

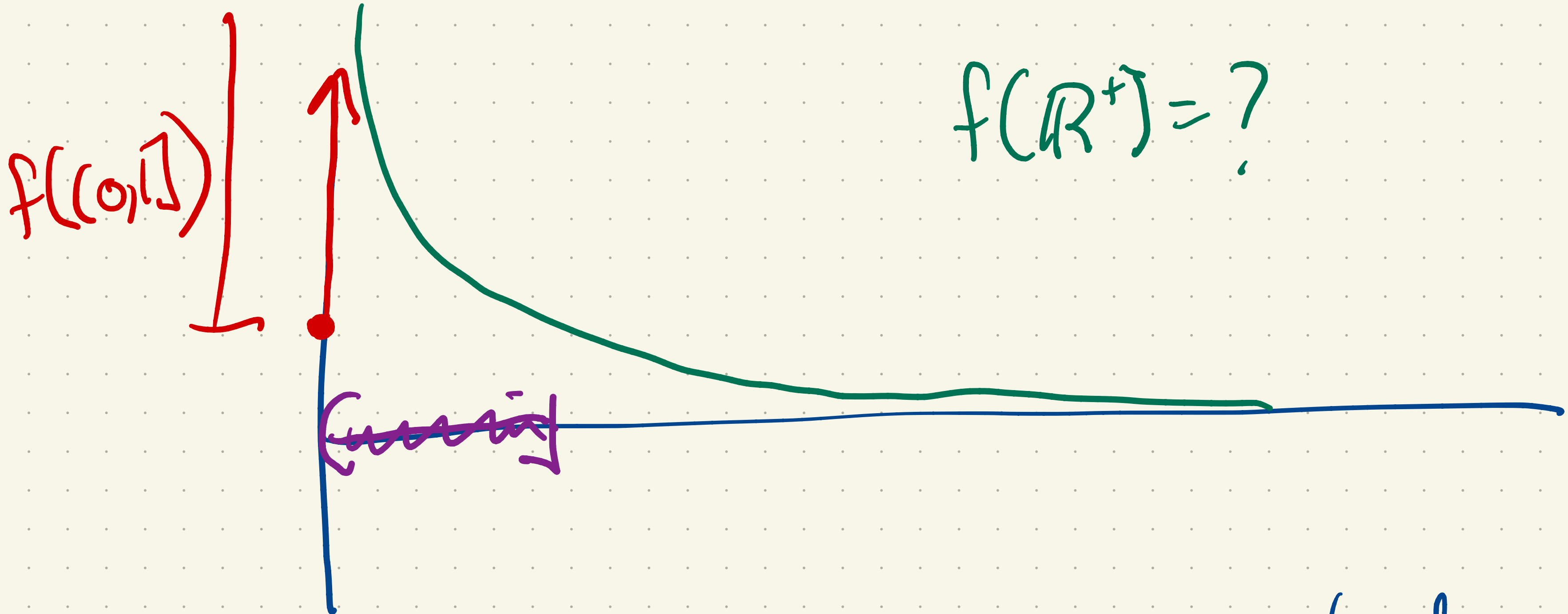
Upshot: has B-W property $\Rightarrow$ is closed + bounded i.e. compact.

The continuous image of a compact set is compact.

$$f : A \to \mathbb{R} \qquad f(A) = \{ f(x) : x \in A \}$$

$$f : \mathbb{R}^+ = \{ x : x > 0 \} \qquad f(x) = \frac{1}{x}$$

$$f(\mathbb{R}^+) = ?$$



$f(\mathbb{R}^+) = ?$

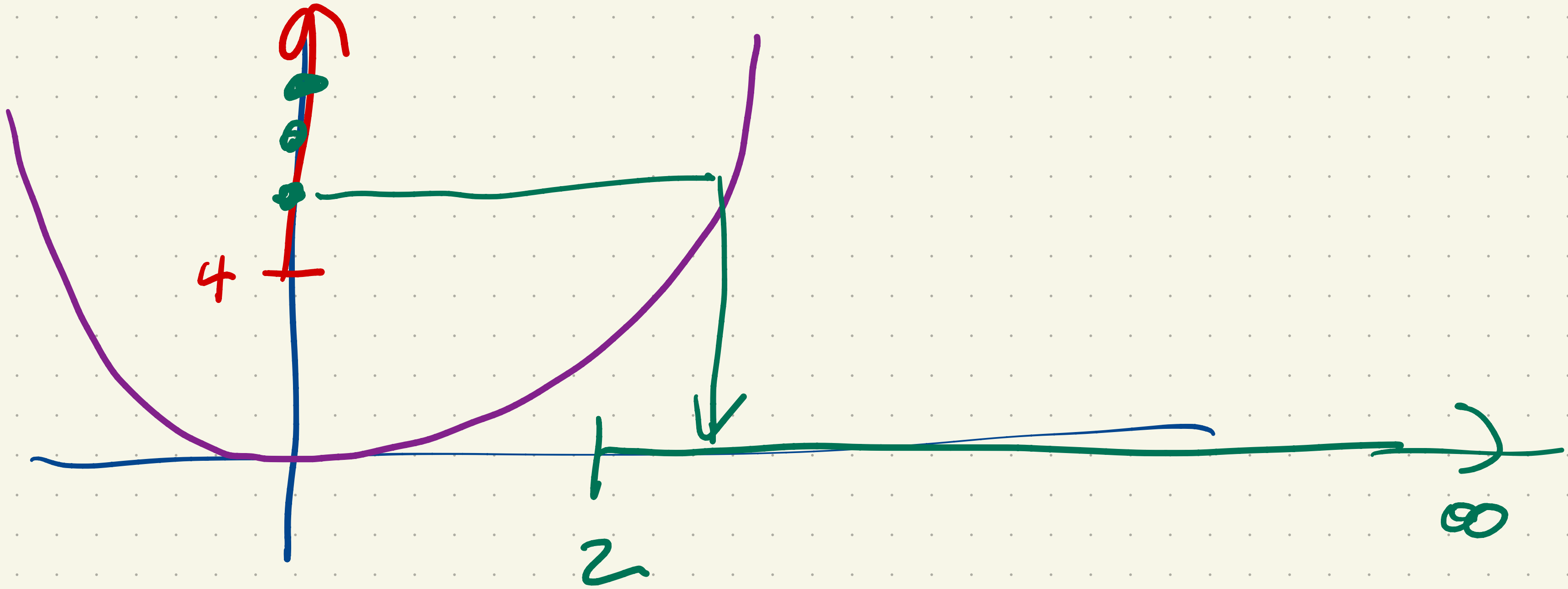
closed + bounded

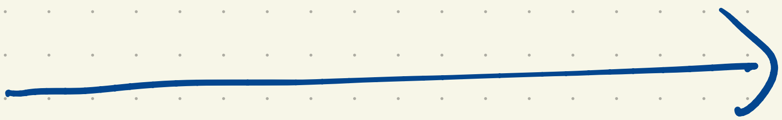
$f: \overbrace{(0,1]}^{A} \to \mathbb{R}$

$f(x) = \frac{1}{x}$ $\qquad$ $f(A) = [1, \infty)$

$A = [2, \infty)$ $\qquad$ $f(A) = [4, \infty)$

$f(x) = x^2$

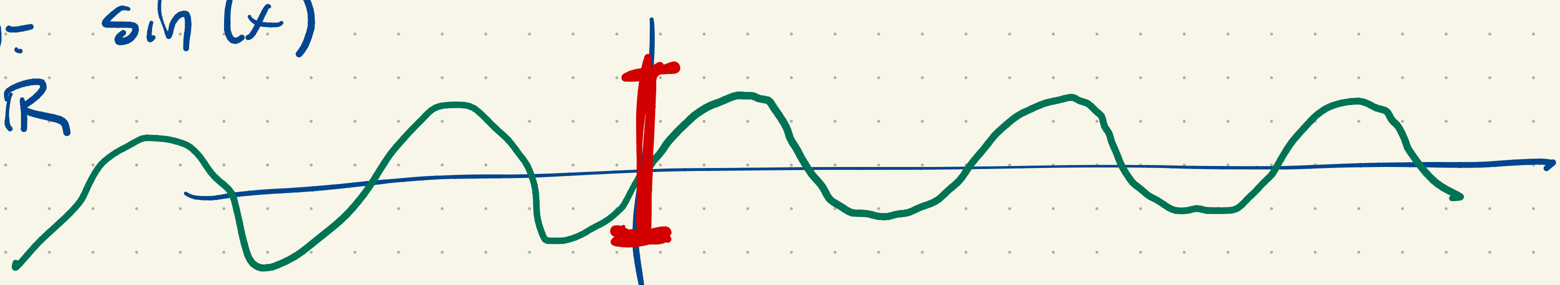




$(-\infty, 2)$

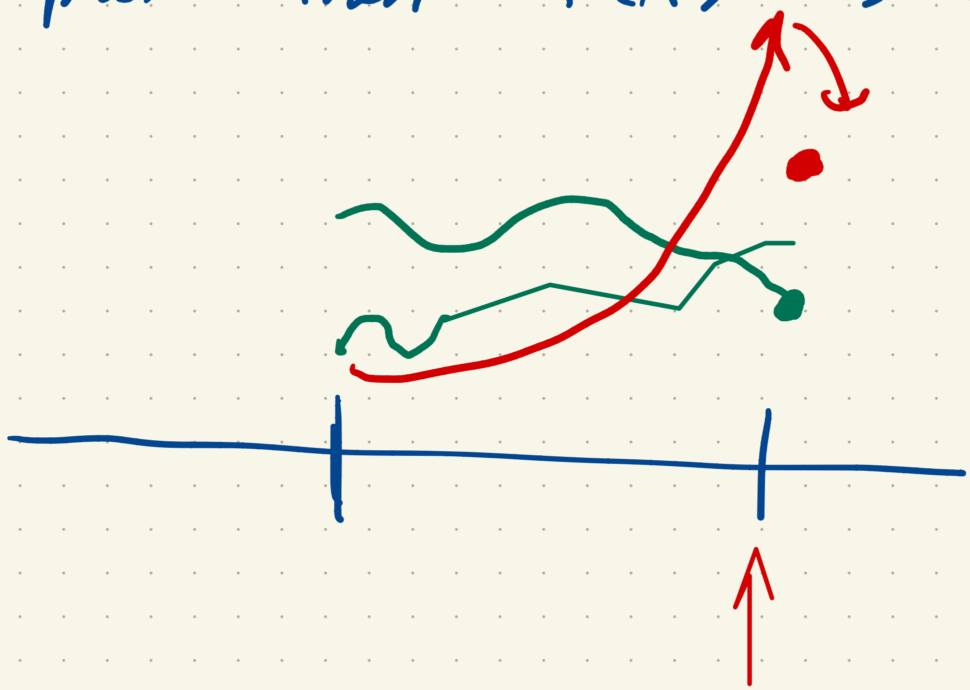
$f(x) = \sin(x)$

$A = \mathbb{R}$



$f(A) = [-1, 1]$

If $f: A \to \mathbb{R}$ is continuous and if A is compact then $f(A)$ is also compact.



Pf: Let (y_n) be a sequence in $f(A)$.

Then for all $n \in \mathbb{N}$ there exists $x_n \in A$ with $f(x_n) = y_n$.

Since A is compact, (x_n) has a subsequence (x_{n_k}) converges to a limit $x \in A$. Observe $f(x) \in f(A)$.

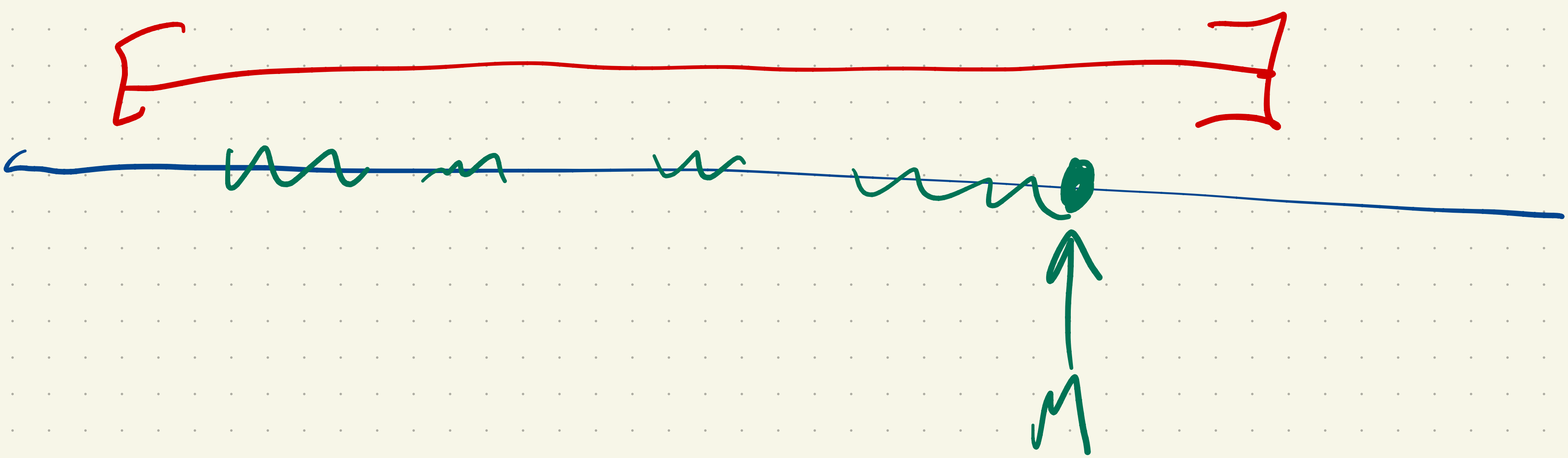
By continuity $f(x_{n_k}) \to f(x)$

$$y_{n_k} \longrightarrow f(x) \in f(A).$$

$\square$

On HW: If $K \subseteq \mathbb{R}$ is compact and $\neq \emptyset$ then there exists $M \in K$ such that

$a \leq M$ for all $a \in K$.



If K is compact then it has a maximum element

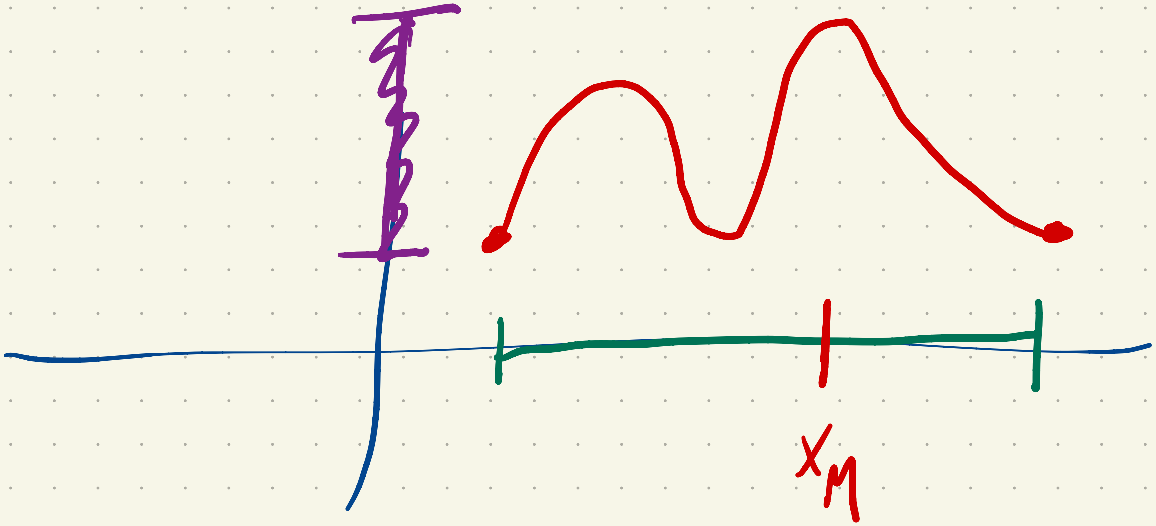
If K is compact and $\neq \emptyset$, then its supremum lies in K.

$$\sup K \in K.$$

Extreme Value Theorem:

Suppose $f: A \to \mathbb{R}$ is continuous where A is compact and $\neq \emptyset$. Then there exists $x_M \in A$ such that

$$f(x) \le f(x_M) \text{ for all } x \in A$$



Pf: Since A is compact and since f is continuous $f(A)$ is compact. By the HW problem there exists $M \in f(A)$ such that $M \geq y$ for all $y \in f(A)$. Since $M \in f(A)$ there exists $x_M \in A$ with $f(x_M) = M$. But then if $x \in A$, $f(x) \in f(A)$ and $f(x) \leq M = f(x_M)$. $\square$

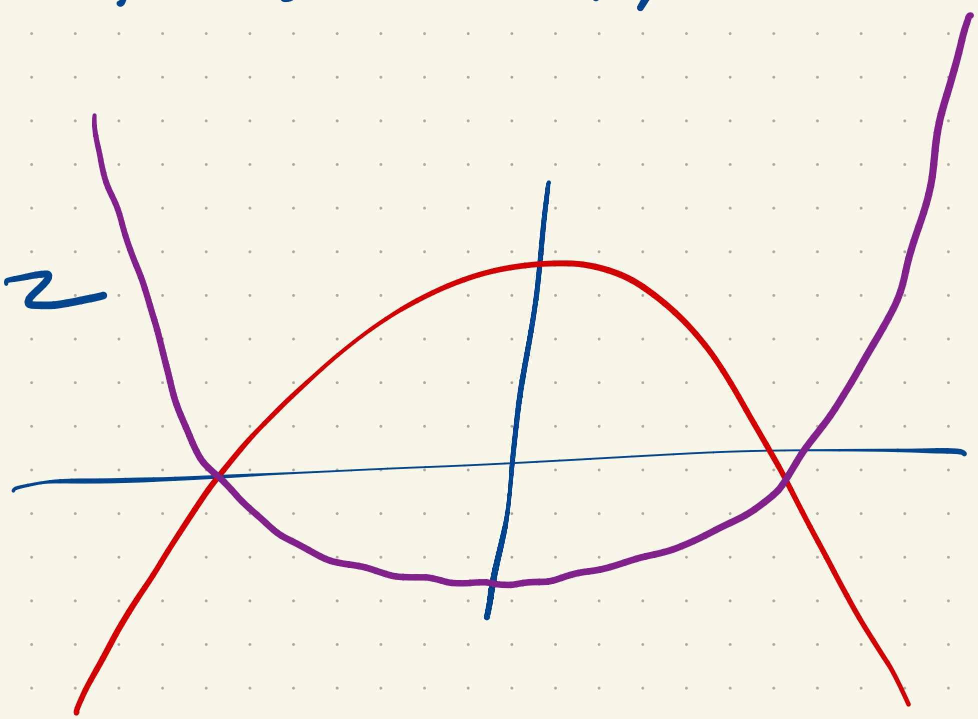
$$g(x) = -f(x)$$

x_m with $g(x) \leq g(x_m)\ \forall\, x \in A$

$$-f(x) \leq -f(x_m)\ \cancel{\forall}\ x \in A$$

$$f(x) \geq f(x_m)\ \forall\, x \in A.$$

$$f(x) = 1 - x^2$$



Uniform Continuity

$f: A \to \mathbb{R}$

continuous

$\forall a \in A$, f is continuous at a

$\forall a \in A$, $\forall \varepsilon > 0$ there exists $\delta > 0$ so

if $x \in A$ and $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

δ depends on both ε and a