Closed set: Contains its limit points.

(c is a limit pt. of A if $\forall \varepsilon > 0$,

$V_\varepsilon(c) \cap (A \setminus \{c\}) \neq \emptyset$.)

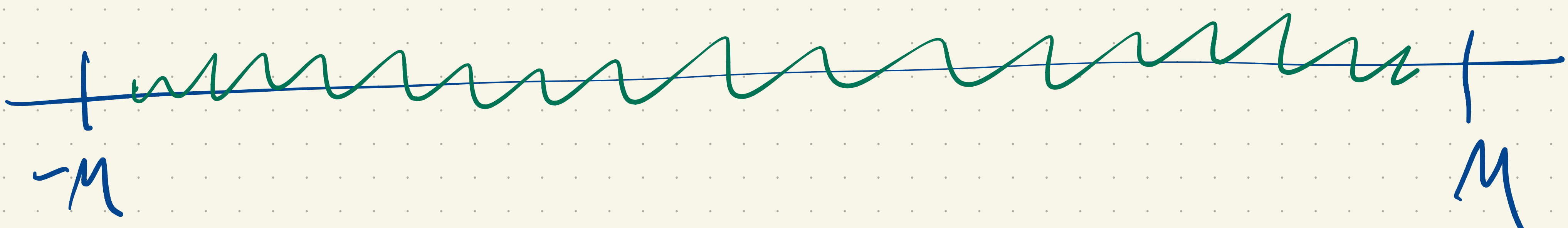
$x_n \in A \setminus \{c\}$

$x_n \to c$

Bounded set: $A \subseteq \mathbb{R}$

There exists some $M \in \mathbb{R}$ such that

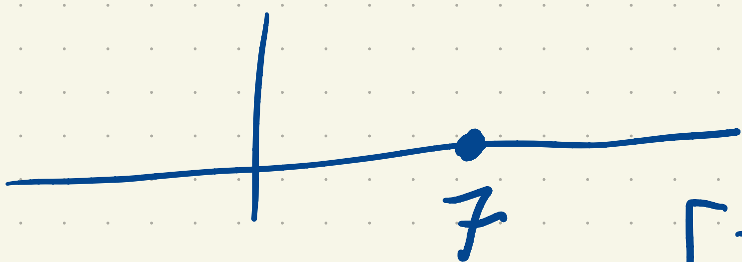
$|a| \leq M$ for all $a \in A$.



Def: A set $K \subseteq \mathbb{R}$ is <u>compact</u> if it is closed and bounded. [Provisional]

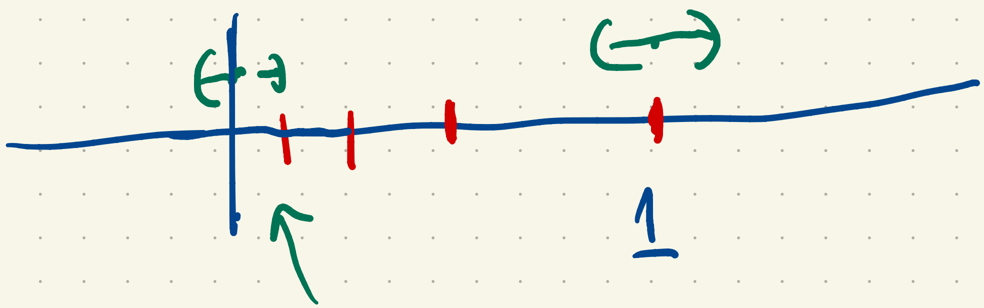
E.g. Closed, bounded intervals are compact.

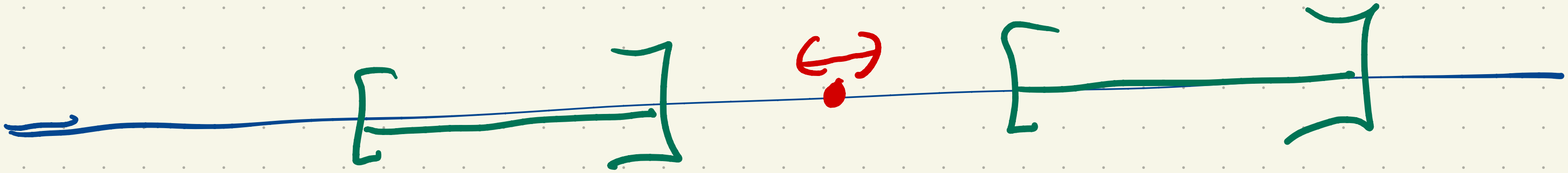
E.g.

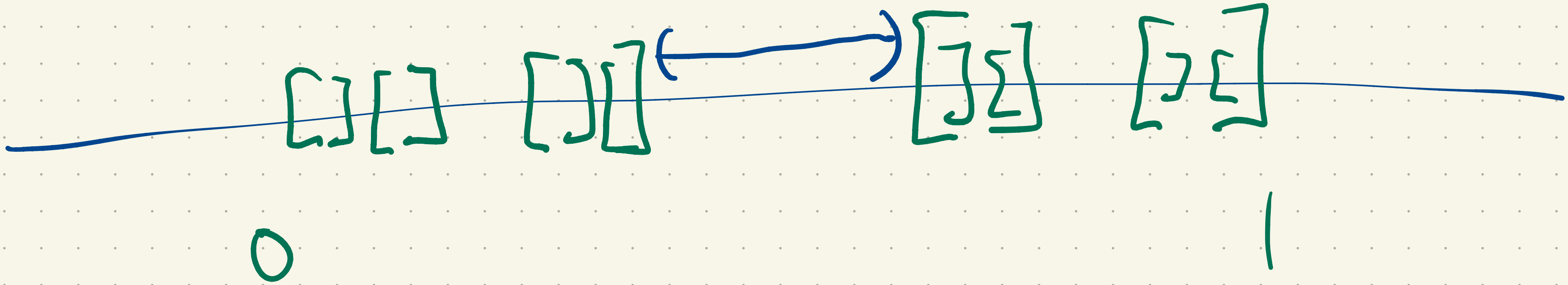


$[7,7]$

$\{\frac{1}{n} : n \in \mathbb{N}\}$





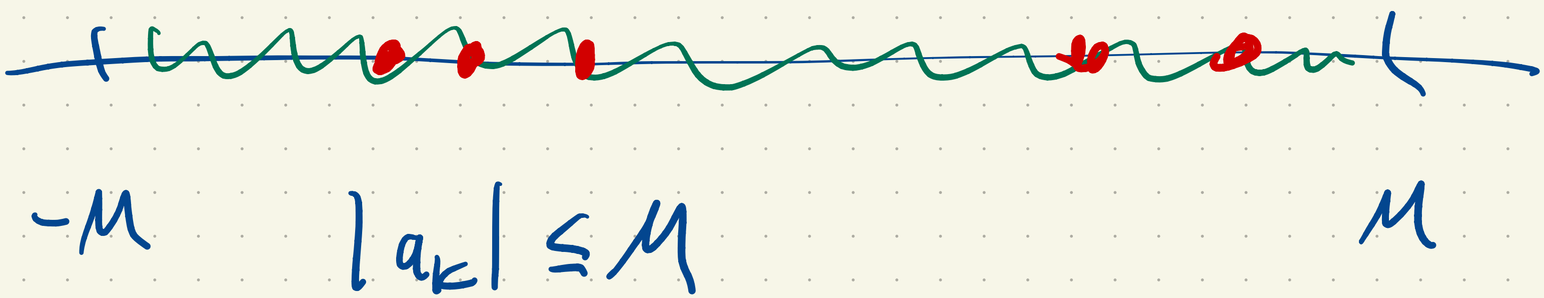


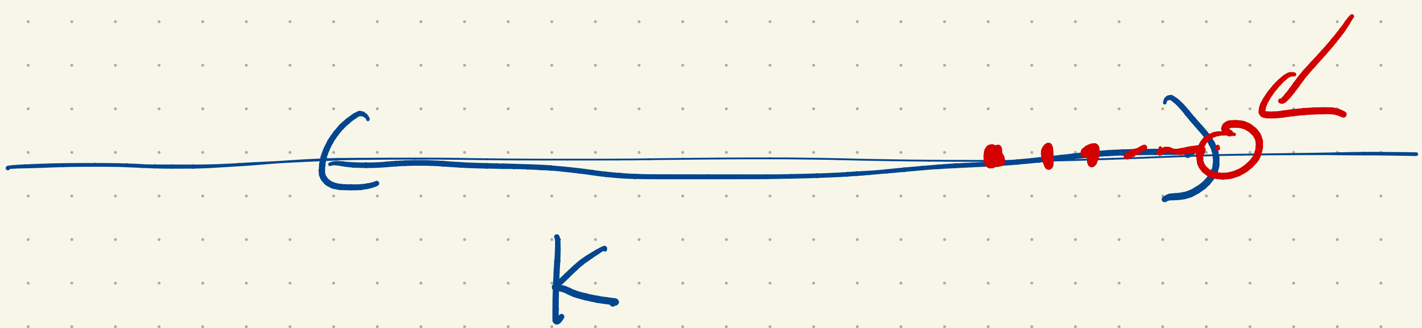
Prop: Suppose $K \subseteq \mathbb{R}$ is compact. Given a sequence (a_k) in K there exists a subsequence (a_{k_j}) that converges to a limit in K.

Pf: Let (a_k) be a sequence in K.

Then since K is bounded the sequence is bounded. By B-W there exists a subsequence (a_{k_i}) converging to a limit L. Since K is closed, $L \in K$.

$\square$





$x_n \in A \longrightarrow x_n \in A \setminus \{c\}$

$x_n \to c$

$c \notin A$

$\Rightarrow$ c is a limit point of A

If A is closed and (x_n) is in A and $x_n \to c$ then $c \in A$.

This property is called the B-W property

of a set: every seq. in the set has a subsequence that converges to a limit in the set

We showed compact sets have the B-W property.

We'll show not compact $\implies$ does not have BW property.

compact $\Leftarrow$ closed <u>and</u> bounded

not compact $\Leftrightarrow$ either not closed <u>or</u> not bounded.

We'll show:

a) Not bounded $\Rightarrow$ does not have BWP

b) Not closed $\Rightarrow$ does not have BWP

a) Not bounded $\Rightarrow$ does not have BWP

$A \subseteq \mathbb{R}$, not bounded.

For all $n \in \mathbb{N}$ there exists $a_n \in A$ with $|a_n| > n$.

$n_1 < n_2 < n_3 < n_4 < \cdots$

$$\underbrace{|a_{n_j}|}_{} > n_j \geq j$$

$\Rightarrow$ not bounded $\Rightarrow$ not convegent

b) Not closed $\Rightarrow$ does not have BWP

A

There exists a limit point c, $c \notin A$.

So there exists a sequence

(x_n) in $A \setminus \{c\}$ with $x_n \to c$.

If A had the BWP then there

would be a subsequence x_{n_k} that converges

to some $a \in A$. But $x_{n_k} \to c$ also

and by uniqueness of limits $a = c$. $\Rightarrow\Leftarrow$

Upshot: A set is compact $\Longleftrightarrow$ it has the B-W property.

Prop: Suppose $f : K \to \mathbb{R}$ is continuous where $K \subseteq \mathbb{R}$ is compact. Then $f(K) = \{ f(a) : a \in K \}$ is compact.

"The continuous image of a compact set is compact."

Pf: (via the B-W property).

Let (y_n) be a sequence in $f(K)$.

Then for each $n \in \mathbb{N}$ there exists $x_n \in K$ with $f(x_n) = y_n$.

Since K has the B-W property there exists a convergent subsequence $x_{n_k} \to a$ for some $a \in K$. By continuity of f $f(x_{n_k}) \to f(a)$. That is

$$y_{n_k} \to f(a) \in f(K).$$

$\square$