But then, by HW, $f(a_n) \to 0$ also.

Alg. Limit Thm

$f, g : A \to \mathbb{R}$

both continuous at $a \in A$.

$f+g \qquad (f+g)(x) = f(x) + g(x)$

$fg \qquad (fg)(x) = f(x) \cdot g(x)$

$f/g\,(x) = \dfrac{f(x)}{g(x)} \qquad (g(x) \neq 0 \ \forall x \in A)$

If f, g are cts at a so are

$f+g$

fg

f/g $\quad (g \neq 0 \text{ on } A)$

To show F is continuous at $a \in A$

show that if (a_n) is a sequence in A

and $a_n \to a$, then $F(a_n) \to F(a)$

Let (a_n) be a sequence in A such that

$a_n \to a$. [Job: show $(fg)(a_n) \to fg(a)$

$\downarrow$

$f(a_n)g(a_n) \to f(a)g(a)$]

Since f is continuous at a, $f(a_n) \to f(a)$,

and similarly $g(a_n) \to g(a)$.

Hence by the Alg. Limit Thm for sequences $f(a_n)g(a_n) \to f(a)g(a)$.

$\square$

Exercise: Show $f(x) = x$ is continuous at each $a \in \mathbb{R}$.

Def: A function $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ is <u>continuous</u> if it is continuous at each point in its domain.

$f(x) = x$ is a continuous function,

$f(x) = x^2$ is a _ _ _ ___ _ _

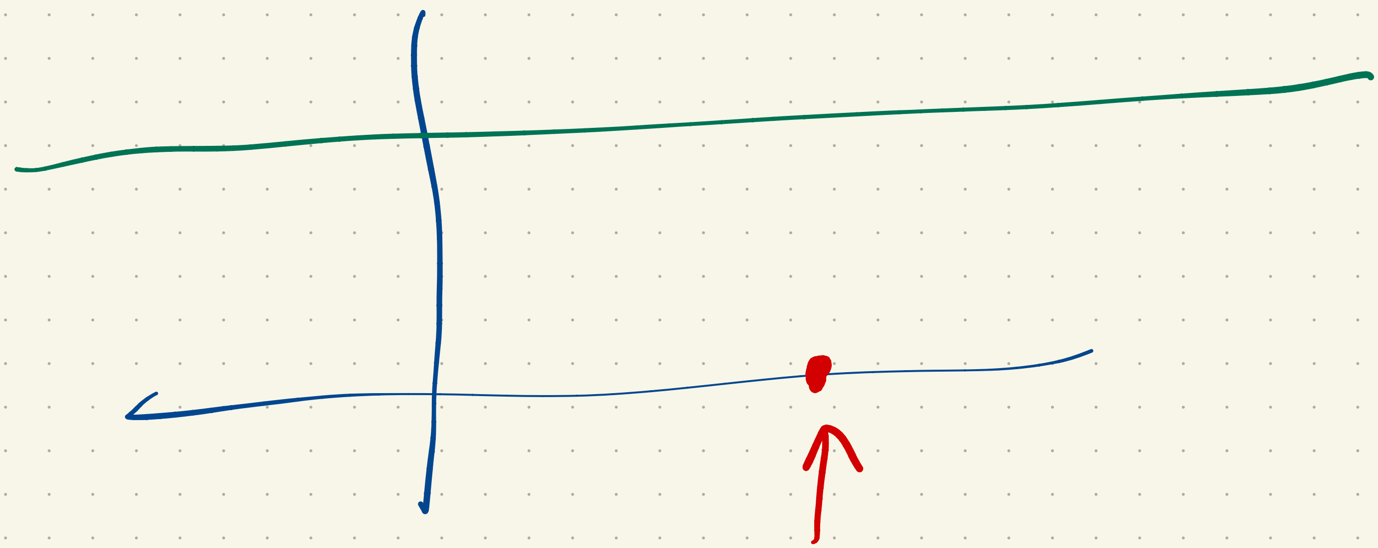
$f(x) = x^k$ is . . . _ _ _ _ _ _ _ . . . $\forall k \in \mathbb{N}$

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

$$c_k \in \mathbb{R} \quad n \in \mathbb{Z}_{\geq 0}$$

$$f(x) = c_1 x$$

$$f(x) = 9$$



$f(x) = \dfrac{p(x)}{q(x)}$ $\quad p, q$ polynomials,

(Domain: $\cancel{p(x)=1 \quad q(x)=x \quad \frac{1}{x}}$

$$\{x : q(x) \neq 0\}$$

Rational functions are continuous on their domains.

Earlier homework: $x_n \geq 0$

$$x_n \rightarrow x \quad (x \geq 0)$$

$$\sqrt{x_n} \rightarrow \sqrt{x}$$

i.e. $\sqrt{\cdot}$ is continuous at each $x \geq 0$.

$$\sqrt{9x^2 + 2x - 6} \rightarrow 9x^2 + 2x - 6 \geq 0$$

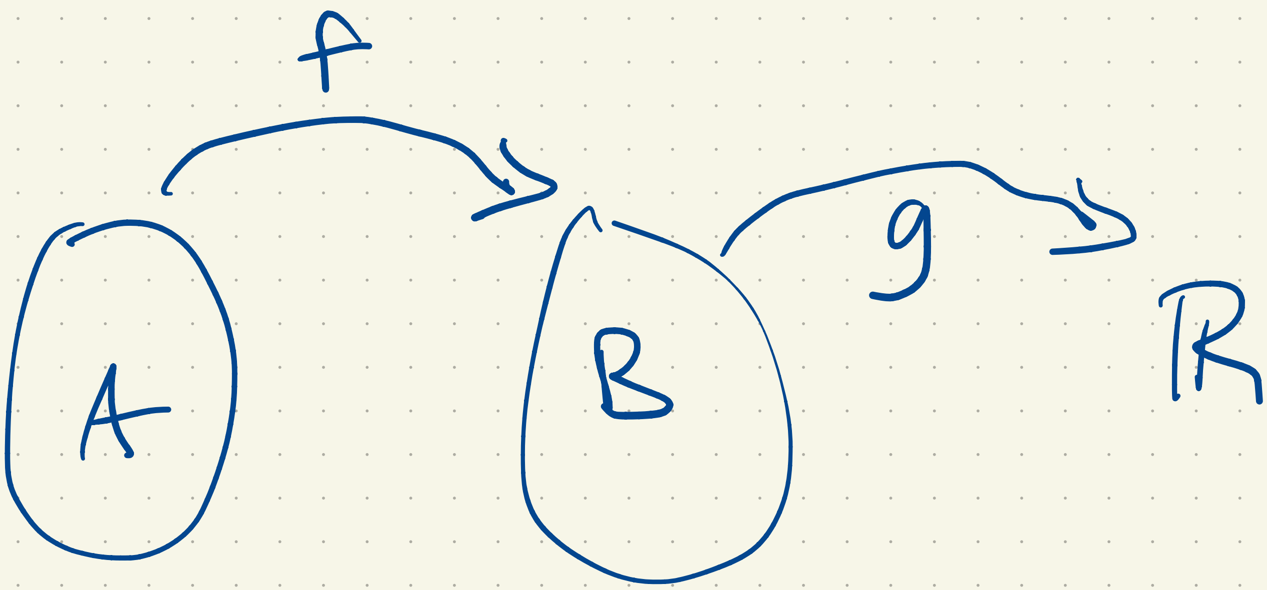
Composition of functions:

Setup: $f : A \to \mathbb{R}$ $\qquad A, B \subseteq \mathbb{R}$

$g : B \to \mathbb{R}$

$f(A) \subseteq B$

$\{ f(a) : a \in A \}$



$$g \circ f : A \to \mathbb{R}$$

$$(g \circ f)(a) = g(f(a))$$

$f: A \to \mathbb{R}$

Prop: Given the setup above, if $g: B \to \mathbb{R}$

f is continuous at $a \in A$, $f(A) \subseteq B$

and g is continuous at $f(a) \in B$

then $g \circ f$ is continuous at a.

Pf: On HW (2 ways, by sequences and by ε-δ).

Want $g \circ f$ is cts at a.

Know f is cts at a

g is cts at $f(a)$.

Need to show that if (a_n) is a sequence in A such that $a_n \to a$,

$(g \circ f)(a_n) \to (g \circ f)(a)$.

Let (a_n) be a sequence in A with $a_n \to a$.

Then $\underbrace{f(a_n)}_{\hookrightarrow b_n} \to f(a)$,

$b_n \to \underbrace{f(a)}_{b} \in B$

$b_n \to b$

g is continuous at $f(a)$

g is continuous at b

$b_n \to b$

$g(b_n) \to g(b)$ | $g(f(a_n)) \to g(f(a))$

$$(g\circ f)(a_n) \to (g\circ f)(a)$$

$$a_n \to a \Rightarrow (g\circ f)(a_n) \to (g\circ f)(a)$$

$\Rightarrow$ $g\circ f$ is continuous at a.

$$g(x) = \sqrt{x}$$

$$\sqrt{9x - 2x + 7}$$

$$B = [0, \infty)$$

$$f(x) = 9x - 2x - 7$$

$$A = \{$$

$$f(A) \subseteq B$$

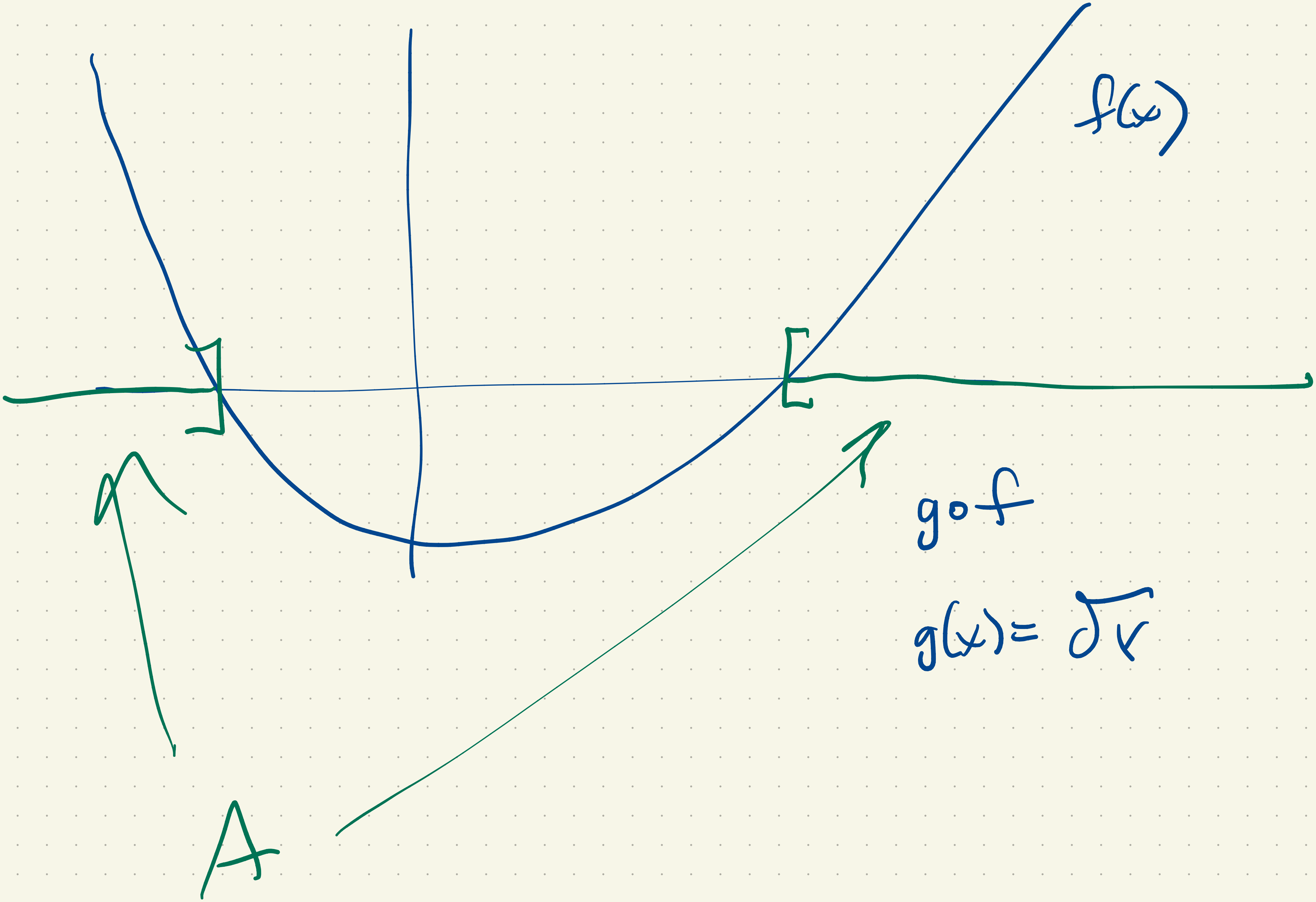
$$A \neq \mathbb{R}$$

$$A \xrightarrow{f} B \xrightarrow{g} \mathbb{R}$$

$$[0, \infty)$$

$$A = \{x \in \mathbb{R} : f(x) \geq 0\}$$

$$= \{x \in \mathbb{R} : f(x) \in B\}$$



$g \circ f$

$g(x) = \sqrt{x}$

Compactness:

$f(x) = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$

$f: A \to \mathbb{R}$

$f(x) = x^3$