a) limit point of a set

b) $\lim_{x \to c} f(x) = L$

c) f is continuous at a.

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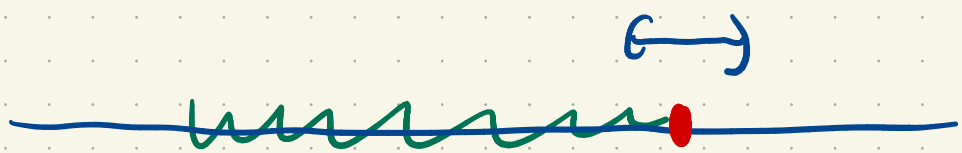
a) $A \subseteq \mathbb{R}$

$c \in \mathbb{R}$

c is a limit point of A



For all $\varepsilon > 0$

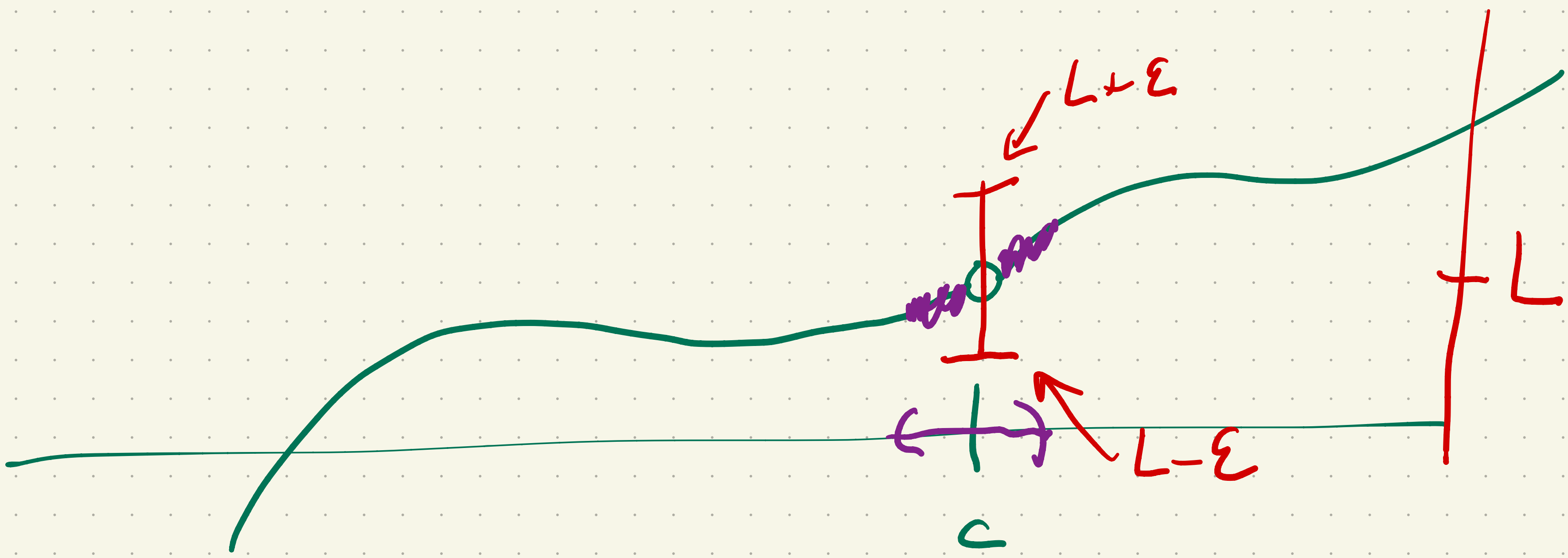
$$V_\varepsilon(c) \cap (A \setminus \{c\}) \neq \emptyset.$$

Every ε-nbhd of c contains a point in A that is not c.

$$\lim_{x \to c} f(x) = L \qquad f : A \to \mathbb{R}$$

c is a limit point of A

For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

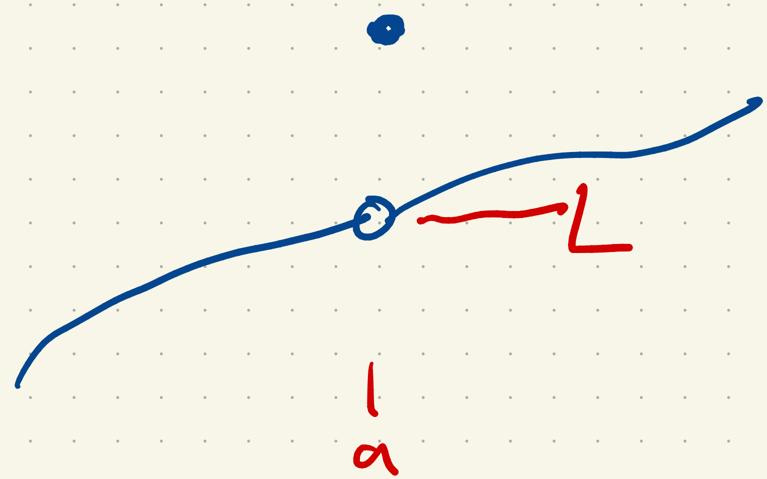


$f : A \to \mathbb{R}$

$a \in A$

f is continuous at $a \in A$

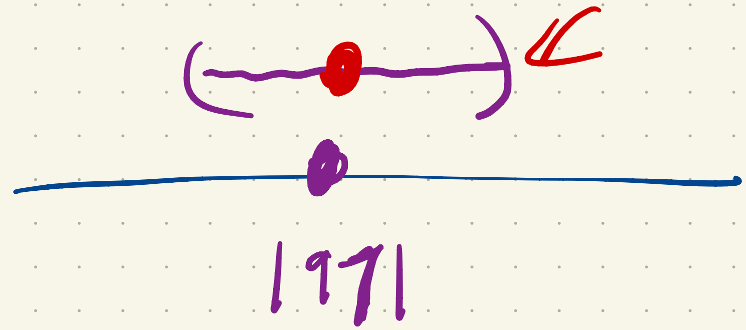
" $\lim_{x \to a} f(x) = f(a)$ "



For all $\varepsilon > 0$ there exists $\delta > 0$ such that

if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Sequential Characterizations



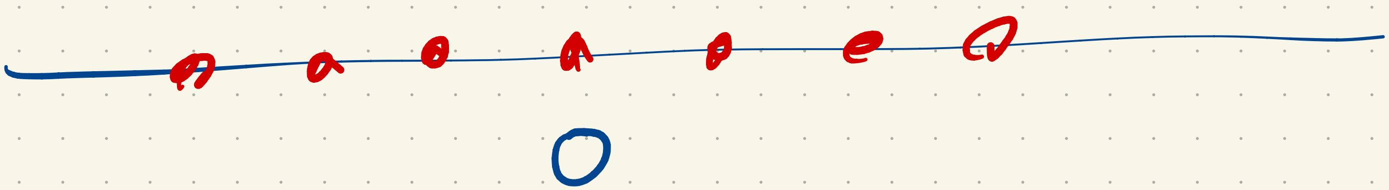
1) c is a limit point of A

$\iff$

there is a sequence (a_n) in $A \setminus \{c\}$

such that $a_n \to c$.

Is 0 a limit point of $\mathbb{Z}$?



There exists a sequence (a_n) in $\mathbb{Z} \setminus \{0\}$ such that $a_n \to 0$.

For all sequences (a_n) in $\mathbb{Z} \setminus \{0\}$,

$$a_n \not\to 0.$$

Consider a sequence (a_n) in $\mathbb{Z} \setminus \{0\}$.

Then for all n, $|a_n - 0| = |a_n| \geq 1$.

But then $a_n \not\to 0$.

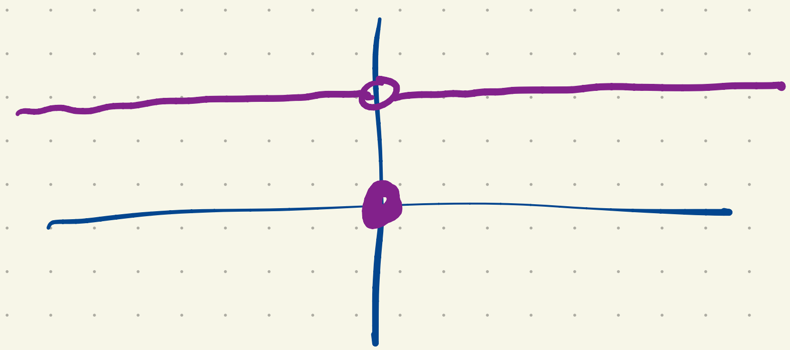
2) $f: A \to \mathbb{R}$

c is a limit point of A

$$\lim_{x \to c} f(x) = L$$

if and only if for all sequences (a_n) in $A \setminus \{c\}$ with $a_n \to c$, $f(a_n) \to L$.

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & \text{otherwise} \end{cases}$$



$$\lim_{x \to 0} f(x) = 1$$

Need to show that for all sequences (a_n) in $\mathbb{R} \setminus \{0\}$ with $a_n \to 0$, $f(a_n) \to 1$.

Suppose (a_n) is a sequence in $\mathbb{R} \setminus \{0\}$ such that $a_n \to 0$. [Job: show $f(a_n) \to 1$].

Since each $a_n \neq 0$, $f(a_n) = 1$ for all n and therefore $f(a_n) \to 1$.

Continuity:

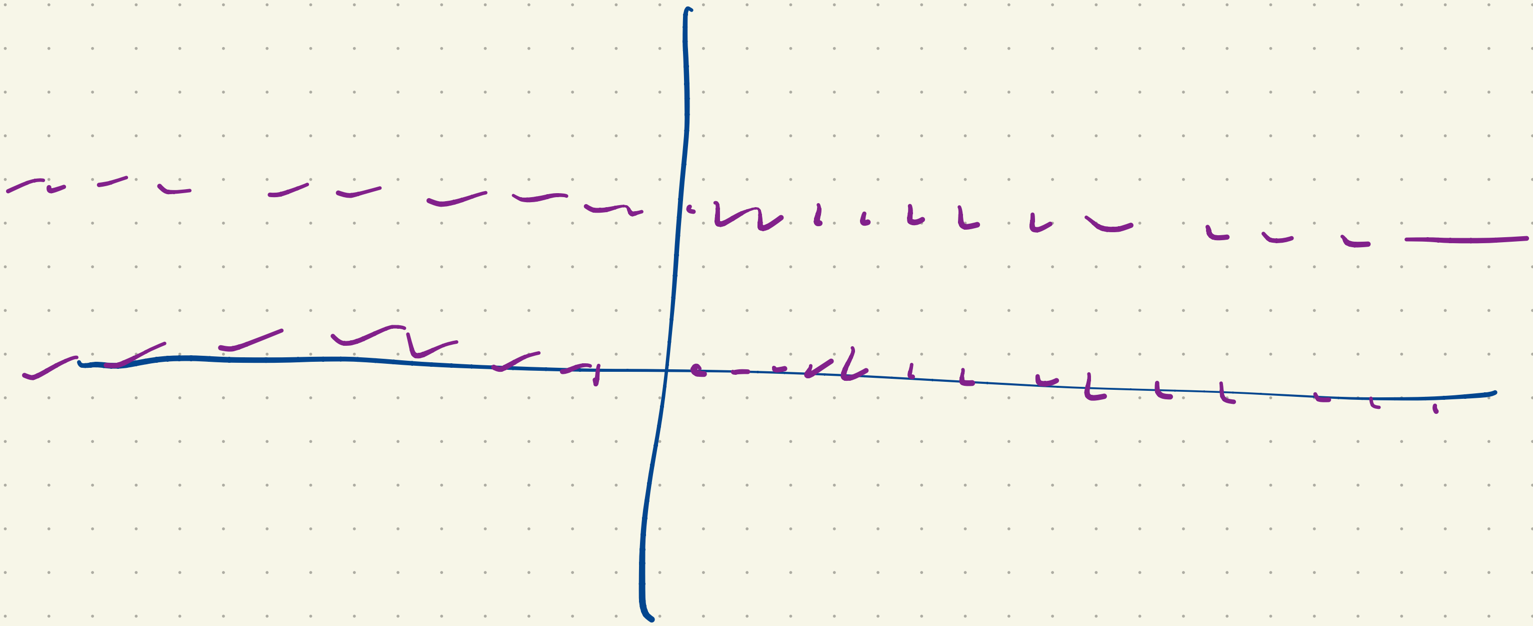
$f : A \to \mathbb{R}$

$a \in A$

f is continuous at a

if and only if for all sequences (a_n) in A with $a_n \to a$, $f(a_n) \to f(a)$.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$



Is this function continuous at $x=0$?

$a_n = \frac{\pi}{n}$ $\qquad a_n \to 0$

$$\begin{cases} f(a_n) = 0 \quad \forall n. \\ f(a_n) \to 0 \\ f(0) = 1 \end{cases}$$

$\Leftarrow$ f is not continuous at 0.

If this function had a limit at 0, it would have to be 1.

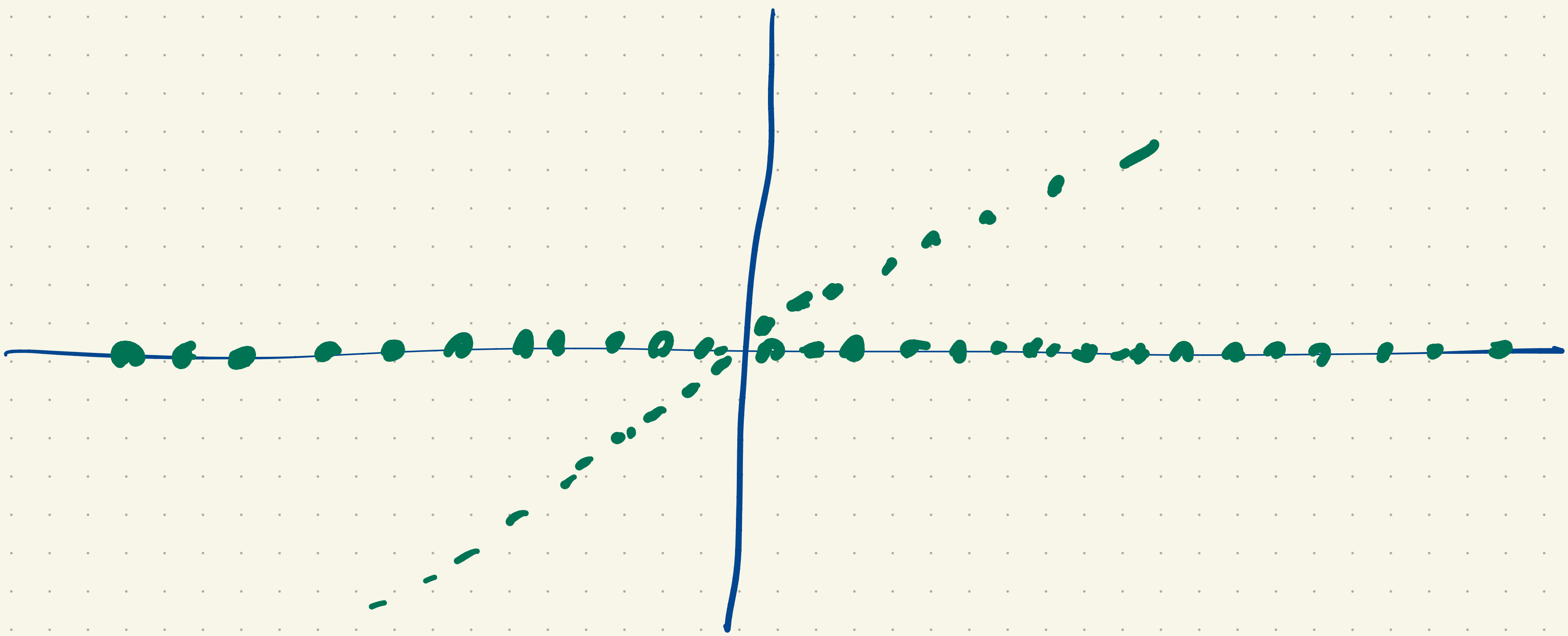
$b_n = \frac{1}{n}$ $\qquad f(b_n) = 1 \quad \forall n.$

$f(b_n) \to 1$

If the limit existed, it would be 1.

Hence $\lim_{x \to 0} f(x)$ does not exist.

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$



— I claim f is continuous at $x=0$.

Using sequences I need to show:

We need to show that if a_n is a sequence with $a_n \to 0$, $f(a_n) \to f(0) = 0$.

Let (a_n) be a sequence in $\mathbb{R}$ with $a_n \to 0$.

Then for all n,

$$0 \leq |f(a_n)| \leq |a_n|$$

Since $a_n \to 0$, $|a_n| \to 0$ so by the squeeze thm, $|f(a_n)| \to 0$.

But then, by HW, $f(a_n) \to 0$ also.