

Continuity:

$$f: A \rightarrow \mathbb{R} \quad A \subseteq \mathbb{R}$$

We say that f is continuous at $a \in A$

if for every $\epsilon > 0$ there exists $\delta > 0$

such that if $x \in A$ and $|x - a| < \delta$

then $|f(x) - f(a)| < \epsilon$.

$$0 < |x - a| < \delta$$

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$\lim_{x \rightarrow c} f(x)$ needs c is a limit point of A

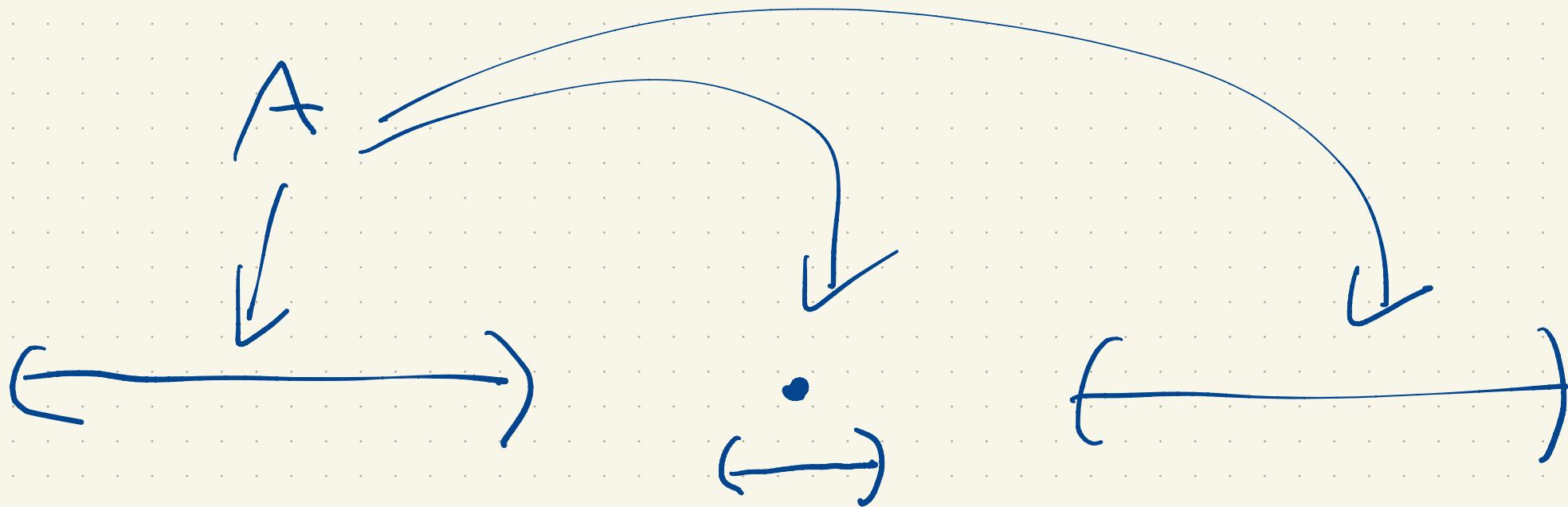
$$A = (0, 1)$$

Prop: Suppose $a \in A$, and a is a limit point of A .

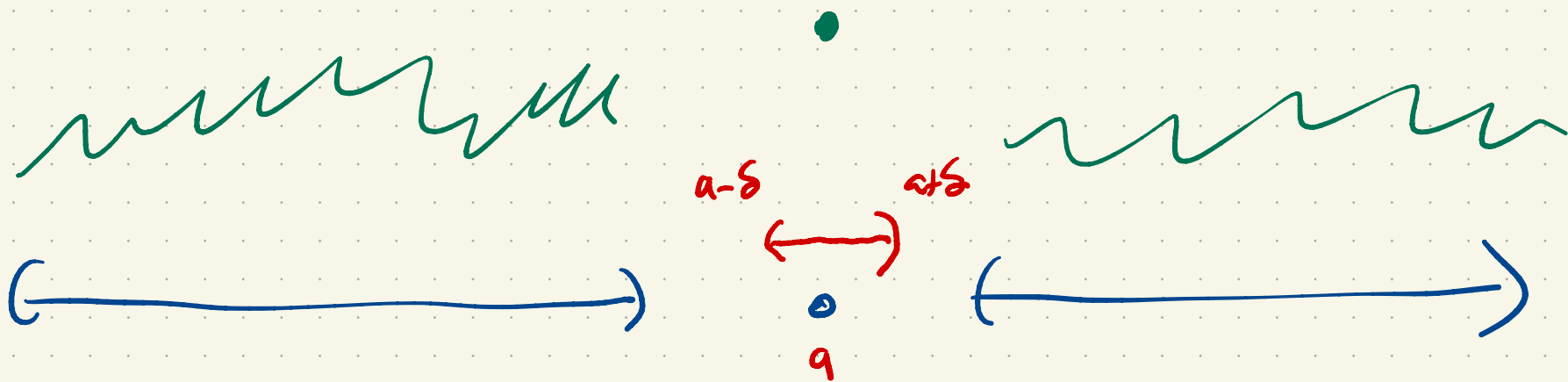
Then f is continuous at a if

and only if

$$\lim_{x \rightarrow a} f(x) = f(a).$$



At isolated points, a function is always continuous.



$$V_\delta(a) \cap A = \{a\}$$

If $x \in A$ and $|x-a| < \delta$

then $x = a$.

$$\underbrace{|f(x) - f(a)|}_{0} < \varepsilon$$

The diagram shows the expression $|f(x) - f(a)| < \varepsilon$. A large bracket underneath the left side of the expression is labeled with a '0'. An upward-pointing arrow is drawn from the '0' to the $f(x)$ term. Another upward-pointing arrow is drawn from the ε term to the right side of the inequality.

Prop: (Sequential Characterization of Continuity)

Suppose $f: A \rightarrow \mathbb{R}$, and $a \in A$.

Then f is continuous at a if and only if
whenever (x_n) is a sequence in A , $x_n \rightarrow a$
then $f(x_n) \rightarrow f(a)$.

[Prove as per functional limits.]

$$f, g: A \rightarrow \mathbb{R}$$

both continuous at $a \in A$.

$f + g$ is continuous at a .

Suppose (x_n) is a sequence in A with

$$x_n \rightarrow a. \quad \text{Then} \quad \begin{aligned} f(x_n) &\rightarrow f(a) \\ g(x_n) &\rightarrow g(a) \end{aligned}$$

$$\text{But then} \quad f(x_n) + g(x_n) \rightarrow f(a) + g(a)$$