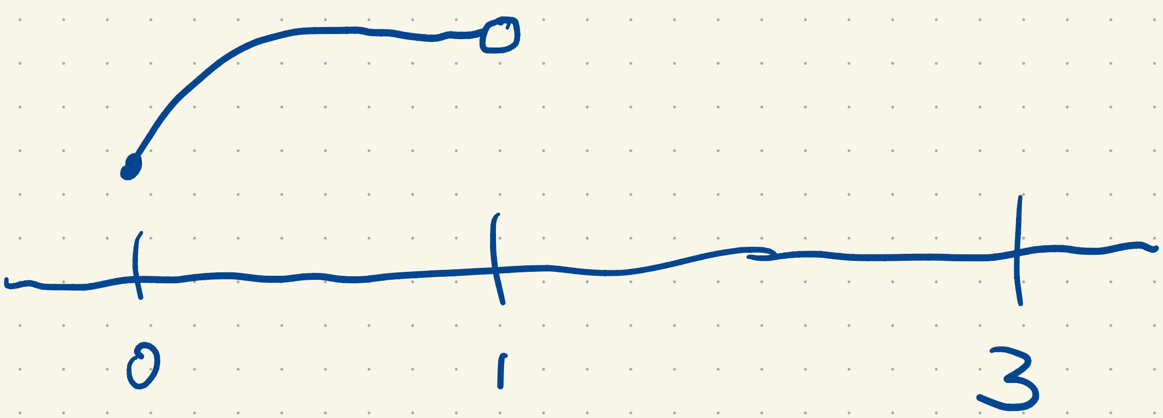
Limits of functions

$A \subseteq \mathbb{R}$ $\qquad f : A \to \mathbb{R}$

What does $\lim_{x \to c} f(x) = L$ mean?

$A = [0,1]$ $\qquad \lim_{x \to 3} f(x)$



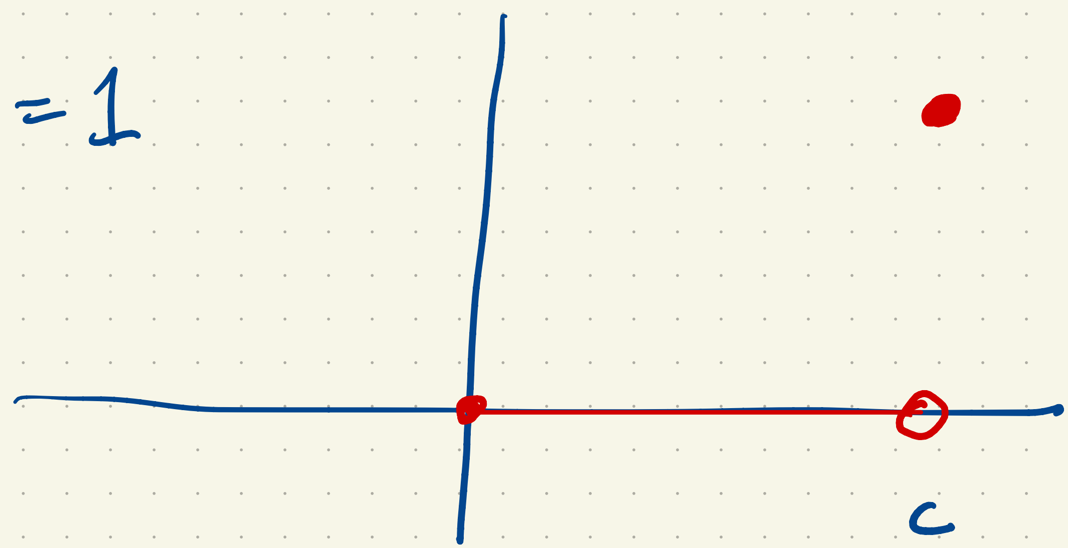
$A = (0,1)$

$$\lim_{x\to 1} f(x)$$

$f:[0,1]\to\mathbb{R}$

$$f(x) = \begin{cases} 0 & x\neq 1 \\ 1 & x=1 \end{cases}$$

$$\lim_{x\to 1} f(x) = \begin{cases} \underline{1} \\ 0 \\ \text{DNE} \end{cases}$$



There exists $\varepsilon_0 > 0$ such that

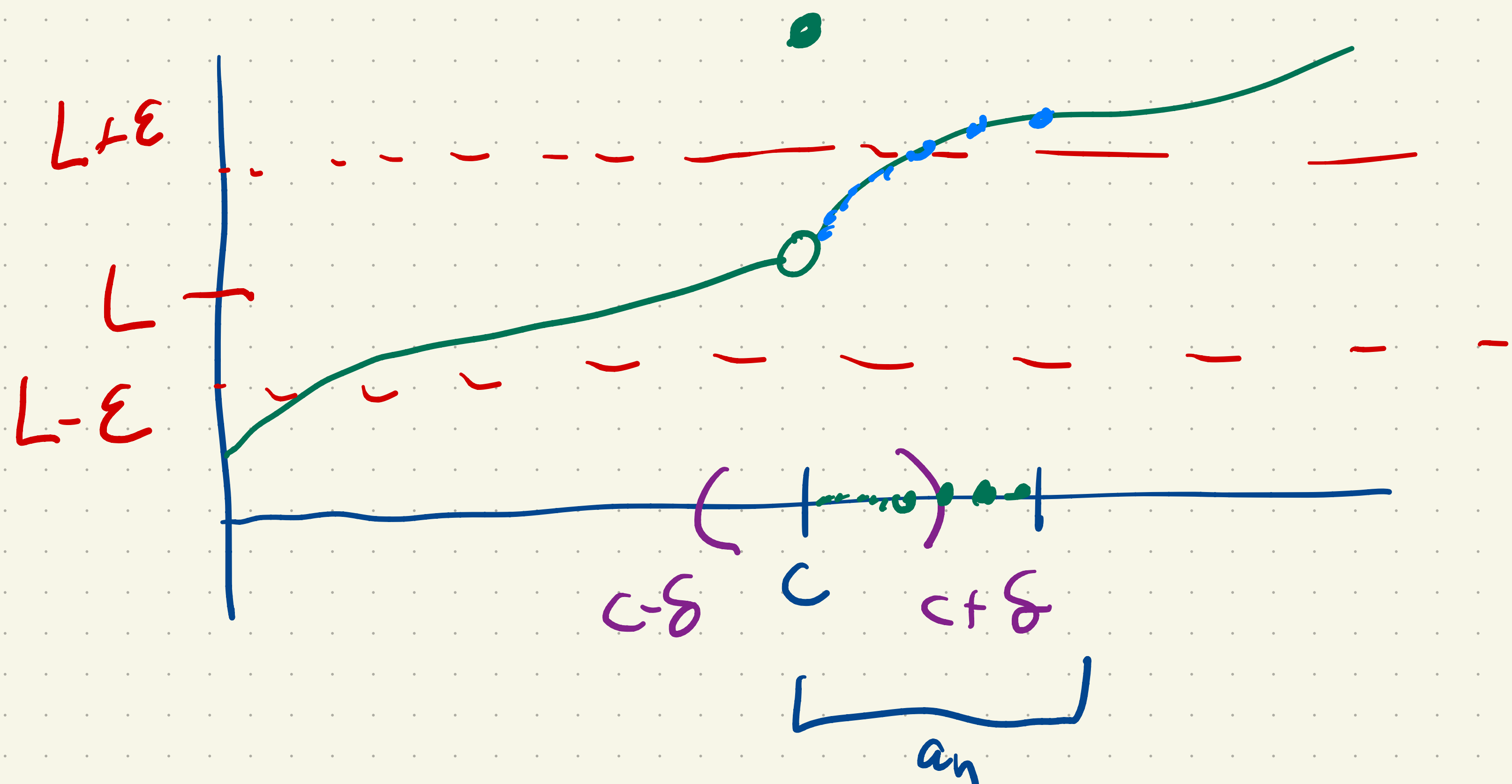
$\forall \delta > 0$ there exists $x \in A$, $0 < |x - c| < \delta$

but $|f(x) - L| \geq \varepsilon_0$.

$0 < |x - c| < \delta$ then

$|f(x) - L| < \varepsilon.$

$A \cap V_\delta(c) \setminus \{c\} \neq \emptyset$



Def: If $f: A \to \mathbb{R}$ and c is a limit point of A we say f diverges at c if $\lim_{x \to c} f(x) \neq L$ $\forall L \in \mathbb{R}$

Lemma: Suppose $f: A \to \mathbb{R}$ and $\lim_{x \to c} f(x) = L$. If (a_n) is a sequence in A with $a_n \neq c$ $\forall n$ then

$$\lim_{n \to \infty} f(a_n) = L.$$

(a_n) $\quad a_n \to c$

$(f(a_n))$

Pf: Let $\varepsilon > 0$. Pick δ so that if $x \in A$ and $0 < |x-c| < \delta$ then $|f(x) - L| < \varepsilon$. Since $a_n \to c$ there exists N so if $n \geq N$

$$0 < |a_n - c| < \delta.$$

Hence if $n \geq N$, $|f(a_n) - L| < \varepsilon$. $\square$

Lemma: Suppose $f: A \to \mathbb{R}$ and c is a limit point of A. If for some $L \in \mathbb{R}$ for every sequence (a_n) in $A \setminus \{c\}$ converging to c, $f(a_n) \to L$, then $\lim_{x \to c} f(x) = L$.

Pf: By contrapositive. Suppose it is not true that $\lim_{x \to c} f(x) = L$. Then there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exists $x \in A$ with $0 < |x - c| < \delta$ and

$$|f(x) - L| \geq \varepsilon_0.$$

$\frac{1}{2^n}$ $\frac{1}{\sqrt{n}}$

Now apply this to each $\delta = \frac{1}{n}$ to get a sequence x_n with

$$0 < |x_n - c| < \frac{1}{n}$$

$x_n \in A \setminus \{c\}$

but $|f(x_n) - L| \geq \varepsilon_0.$

By the squeeze theorem $x_n \to c.$

But then $f(x_n) \not\to L$ as $|f(x_n) - L| \geq \varepsilon_0$

for all n.

Sequential Characterization of Function Limits.

Thm: Let $f: A \to \mathbb{R}$ and let c be a limit point of A. Then $\lim_{x \to c} f(x) = L$ iff

for all sequences (a_n) in $A \setminus \{c\}$

$$\lim_{n \to \infty} f(a_n) = L.$$

Prop: Suppose $f(x) \geq 0$ for all $x \in A$.

If $\lim_{x \to c} f(x) = L$ then $L \geq 0$.

Pf: Consider a sequence (a_n) in A

with $a_n \to c$, $a_n \neq c$ $\forall n$.

Then $f(a_n) \to L$. But $f(a_n) \geq 0$

for all n so by the Limit Order Thm

$L \geq 0$.

e.g. $f(h) = \dfrac{(1+h)^2 - 1}{h}$

$$\lim_{h \to 0} f(h)$$

$a_n \to 0 \qquad a_n \neq 0 \ \forall n$

$$f(a_n) = \frac{(1+a_n)^2 - 1}{a_n} = \frac{1 + 2a_n + a_n^2 - 1}{a_n}$$

$$= 2 + a_n \to 2$$

$$\Rightarrow \lim_{h \to 0} f(h) = 2. \qquad \left[\frac{d}{dx} x^2 \Big|_{x=1} = 2 \right]$$

Alg. Limit Theorem for Functions

$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M \qquad f, g : A \to \mathbb{R}$

c: limit pt of A

a) $\lim_{x \to c} a f(x) = aL$

b) $\lim_{x \to c} f(x) + g(x) = L + M$

c) $\lim_{x \to c} f(x) g(x) = LM$

d) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ [if $M \neq 0$ and if $g(x) \neq 0$ on A]

Divergence Criteria:

Suppose $f: A \to \mathbb{R}$ and c is a limit point of A.

If either

1) There exists a sequence (a_n) in $A \setminus \{c\}$ such that $(f(a_n))$ does not converge

or

2) There exist two sequences (a_n), (b_n) in $A \setminus \{c\}$ with $a_n \to c$ and $b_n \to c$

such that $f(a_n) \to L$ and $f(b_n) \to M$

with $L \neq M$

then f diverges at c.

$f(x) = \frac{1}{x}$

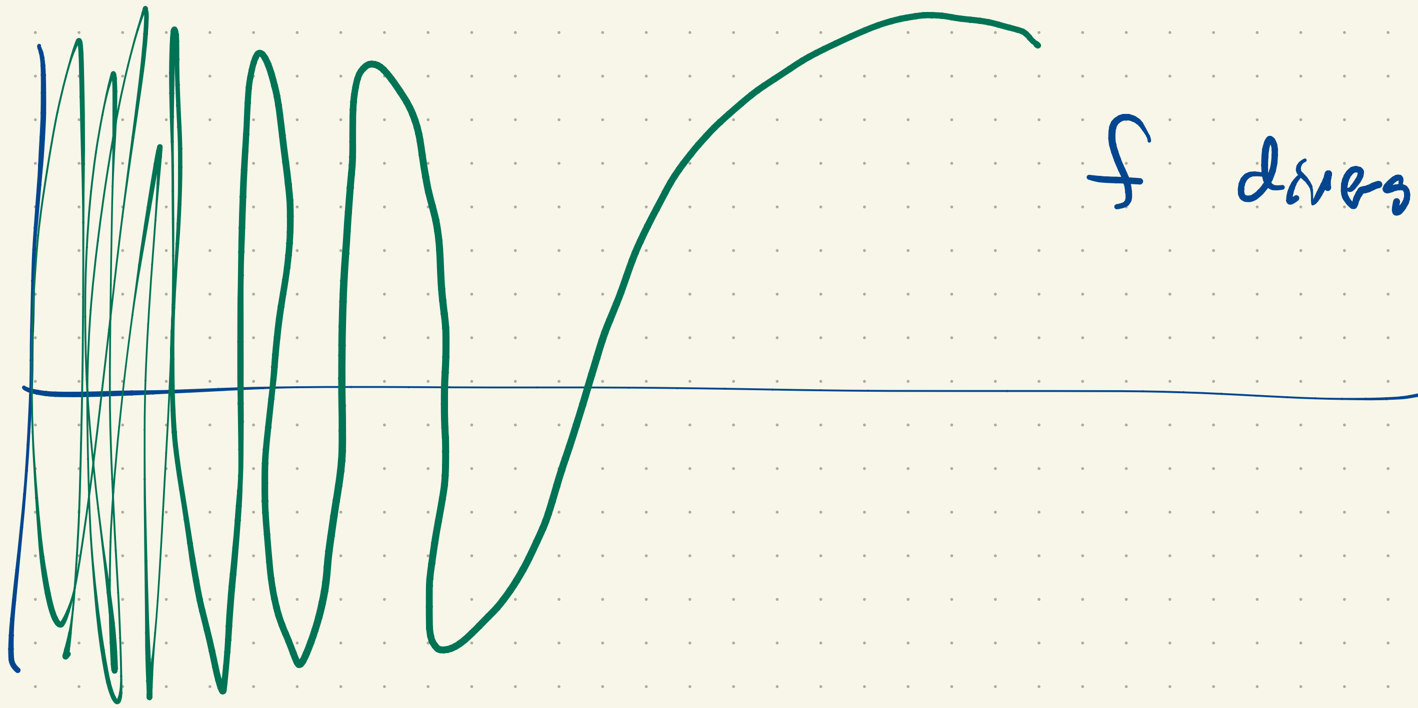
f diverges at 0

$a_n = \frac{1}{n}$ $a_n \to 0$

$a_n \neq 0$

$f(a_n) = n$

$f(x) = \sin\left(\frac{1}{x}\right)$



f diverges at 0