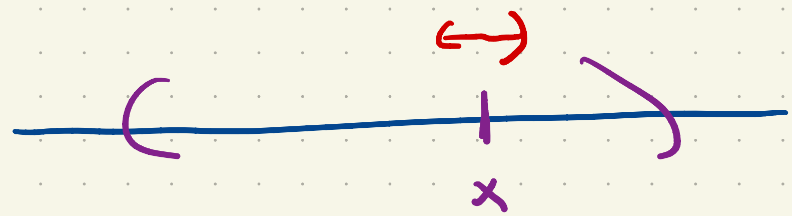
Open set ($A \subseteq \mathbb{R}$)

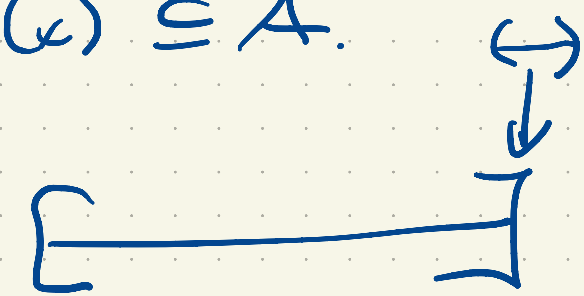
$$V_\varepsilon(x) = (x-\varepsilon,\ x+\varepsilon)$$

$$= \{y \in \mathbb{R} : |x-y| < \varepsilon\}$$



A is open if $\forall x \in A$

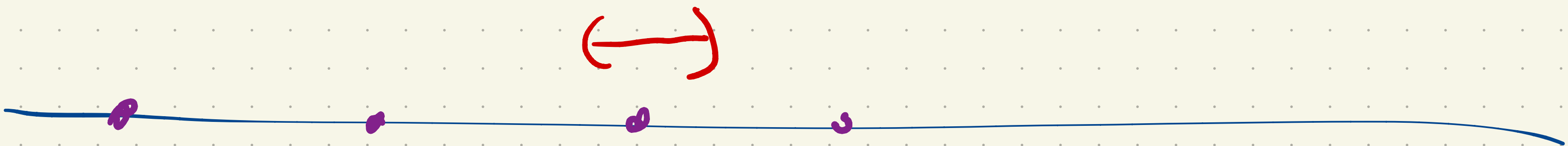
$\exists\ \varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq A$.



Limit point of A:

x s.t. for all $\varepsilon > 0$

$$V_\varepsilon(x) \cap \left(A\setminus\{x\}\right) \neq \emptyset$$



$x \in \mathbb{R}$ is a limit point of A

$\Leftrightarrow$

there is a sequence in $A\setminus\{x\}$ converging to x.

$(0,1)$ $\quad$ 0 is a limit point

since $\frac{1}{n} \to 0$ and

each $\frac{1}{n} \in A \setminus \{0\}$.

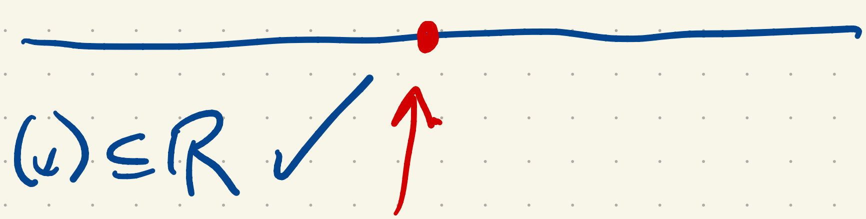
$= A$ $\quad$ η

Def: A set $A \subseteq \mathbb{R}$ is <u>closed</u> if it contains its limit points.

$(0,1)$

$\mathbb{R} \to$ open?

$\forall x \in \mathbb{R},\ V_1(x) \subseteq \mathbb{R}$ ✓



$\mathbb{R}$ closed?

$\mathbb{Z}$

$\emptyset \subseteq \mathbb{Z}$

$\emptyset \to$

open set
closed

Prop. If $A \subseteq \mathbb{R}$ is closed and (x_n) is a seq. in A converges to some limit L, $L \in A$.

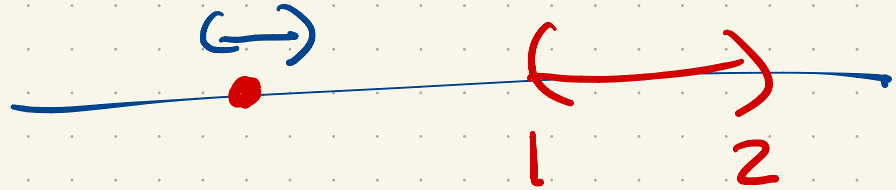
Pf: Suppose to the contrary that A is closed and (x_n) is a seq. in A converges to a limit $L \notin A$. Then, since $L \notin A$, (x_n) is a sequence in $L \setminus \{A\}$ converges to L. Hence L is a limit pt. of A. Since A is closed $L \in A$.

$\square$

Prop: Suppose $A \subseteq \mathbb{R}$ has the property that every convergent sequence in A converges to a limit in A. Then A is closed.

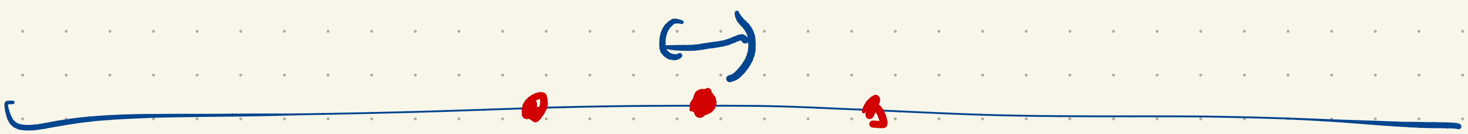
Pf: Let L be a limit point of such a set A. Then there exists a sequence in $A \setminus \{L\}$ converging to L. Such a sequence is a sequence in A. Thus $L \in A$ by hypothesis and A is closed. $\square$

$A = \{0\} \cup (1,2)$



point in A
not a limit point

isolated point.



Closed sets: a) contain their limit point

b) closed under taking sequences

c) complements of open set

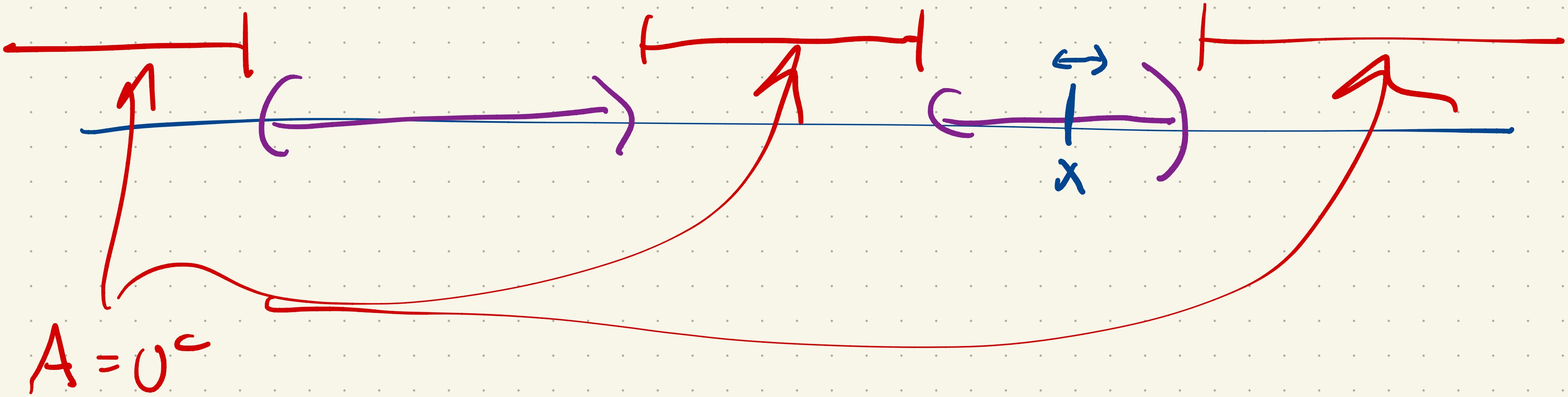
Prop: Suppose $U \subseteq \mathbb{R}$ is open. Then U^c is closed.

pf: Let $A = U^c$. Consider some $x \in A^c = U$.

Since U is open there exists $\varepsilon > 0$ such that

$V_\varepsilon(x) \subseteq U$. So $V_\varepsilon(x) \cap U^c = \phi$.

That is $V_\varepsilon(x) \cap A = \phi$. So x is not a limit pt. of A.



If $x \notin A \Rightarrow x$ is not a limit pt of A.

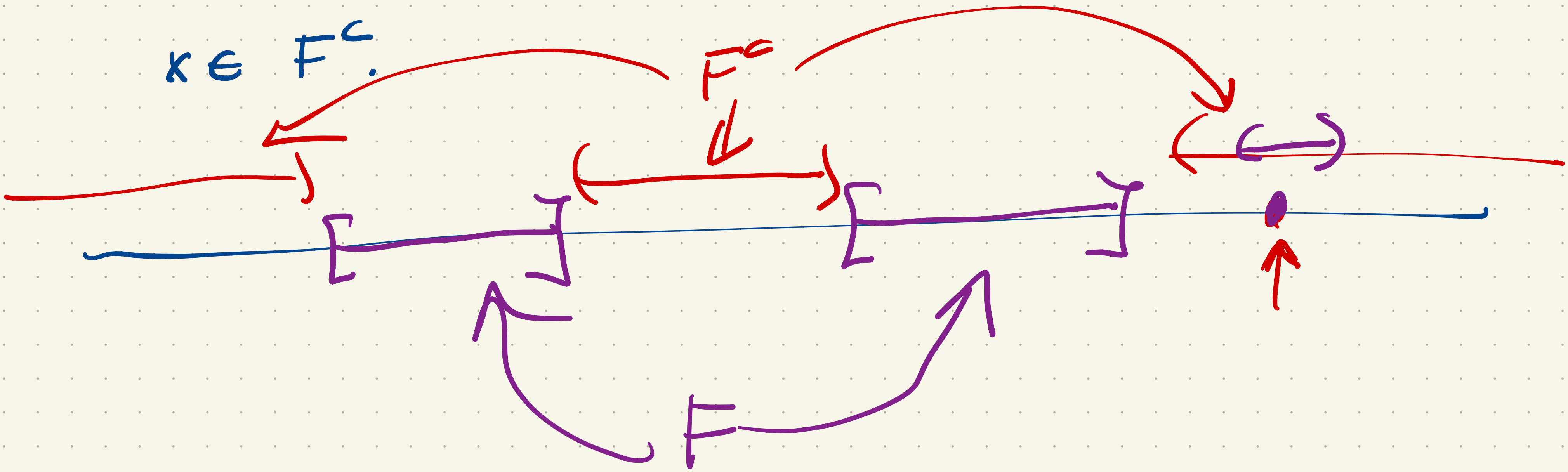
Hence A is a set that contains its limit points and is hence closed.

$\square$

Prop: Suppose F is closed. Then F^c is open.

Pf: Suppose F is closed and consider

$x \in F^c$.



Since F contains its limit points x is not a limit point.

Thus there exists $\varepsilon > 0$ such that

$V_\varepsilon(x) \cap F \subseteq \{x\}$. But $x \notin F$, so $V_\varepsilon(x) \cap F = \emptyset$.

That is, if $x \in F^c$ there is an $\varepsilon > 0$ such that $V_\varepsilon(x) \cap F = \phi$ and therefore $V_\varepsilon(x) \subseteq F^c$.

So F^c is open.

$\square$

$\mathbb{R}$: open $\qquad \mathbb{R}^c = \phi$
closed

1:30 $\qquad$ 1-2