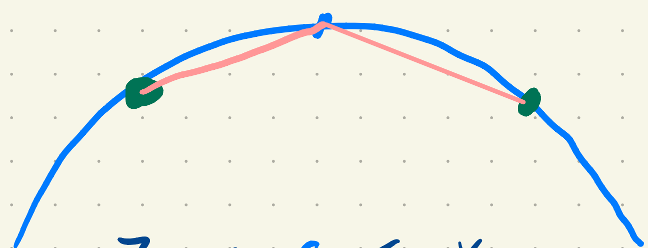
$f \mapsto -f$

**Fermat's Theorem:**

Suppose $f:[a,b]\to\mathbb{R}$ attains a maximum at $c\in(a,b)$ and $f$ is differentiable at $c$. Then $f'(c)=0$.

$$f'(c)=\lim_{x\to c}\frac{f(c)-f(x)}{c-x}$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$



$z_n \to c \leftarrow x_n$

$x_n\to c \quad x_n\neq c \ \forall n.$

$$f(c)-f(x_n)\geq 0$$

$$c-x_n<0$$

$$\lim_{n\to\infty} \underbrace{\frac{f(c)-f(x_n)}{c-x_n}}_{\leq 0} = \lim_{x\to c} \frac{f(c)-f(x)}{c-x}$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(c)-f(x_n)}{c-x_n} \leq 0.$$

$f'(c) \leq 0$

$f'(c) \geq 0$

$\Rightarrow f'(c) = 0$

Pf: Since $c \in (a,b)$ there exists a sequence $(x_n)$ in $[a,b]$ with $c < x_n$ for all $n$ and $x_n \to c$.

Observe that since $f(c) \geq f(x_n)$ for all $n$,

$$\frac{f(c) - f(x_n)}{c - x_n} \leq 0$$

$c < x_n$ and since

for all $n$.

But then

$$f'(c) = \lim_{x \to c} \frac{f(c) - f(x)}{c - x} = \lim_{n \to \infty} \frac{f(c) - f(x_n)}{c - x}$$

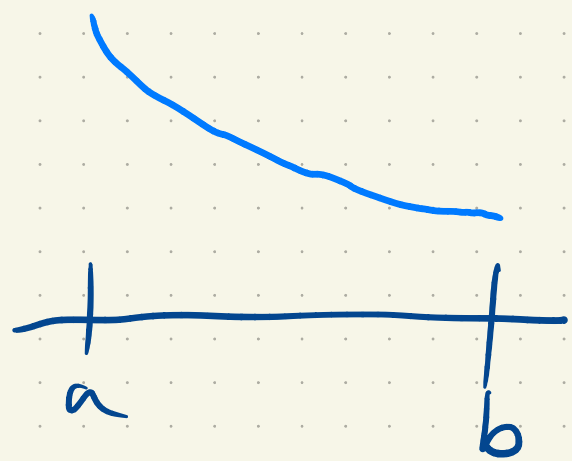
$$\leq 0$$

by the Limit order theorem. A similar proof using a sequence $z_n \to c$ with $z_n < c$ for all $n$ shows $f'(c) \geq 0$ as well and hence $f'(c) = 0$.

$\square$

What if $f$ achieves a max at $a$ and is diff at $a$?

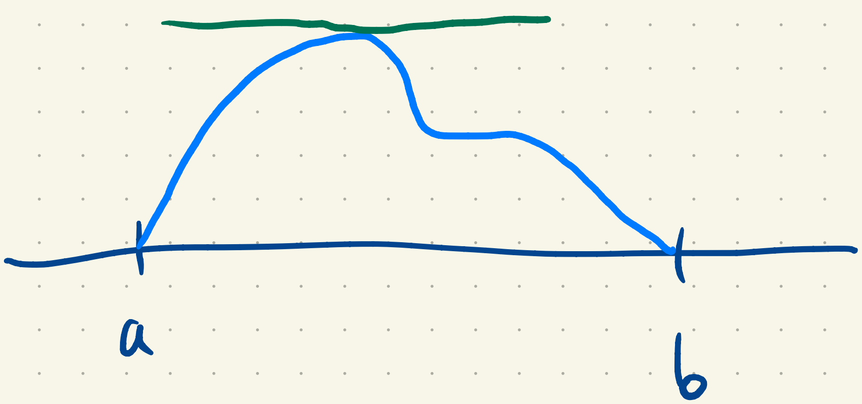
$f'(a) \leq 0$

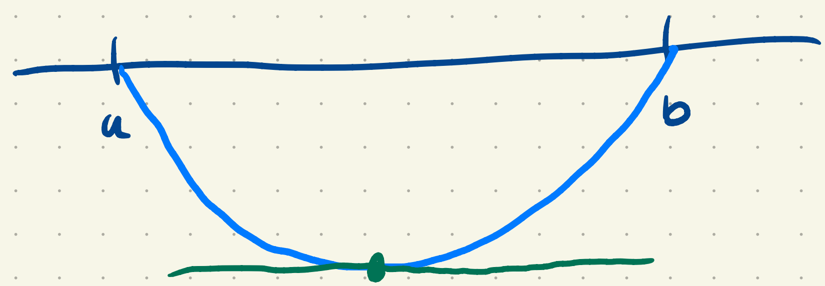


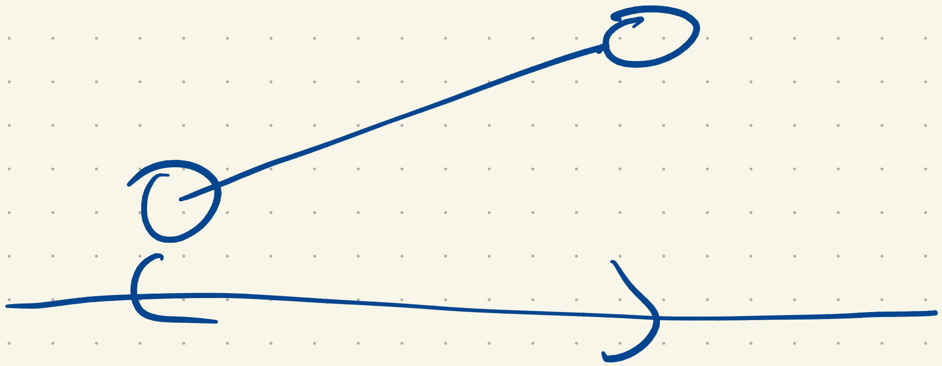

$f'(b) \geq 0$

## Cor (Role's Lemma)

Suppose $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ and $f(a)=f(b)$. Then there exists $c \in (a,b)$ such that $f'(c)=0$.



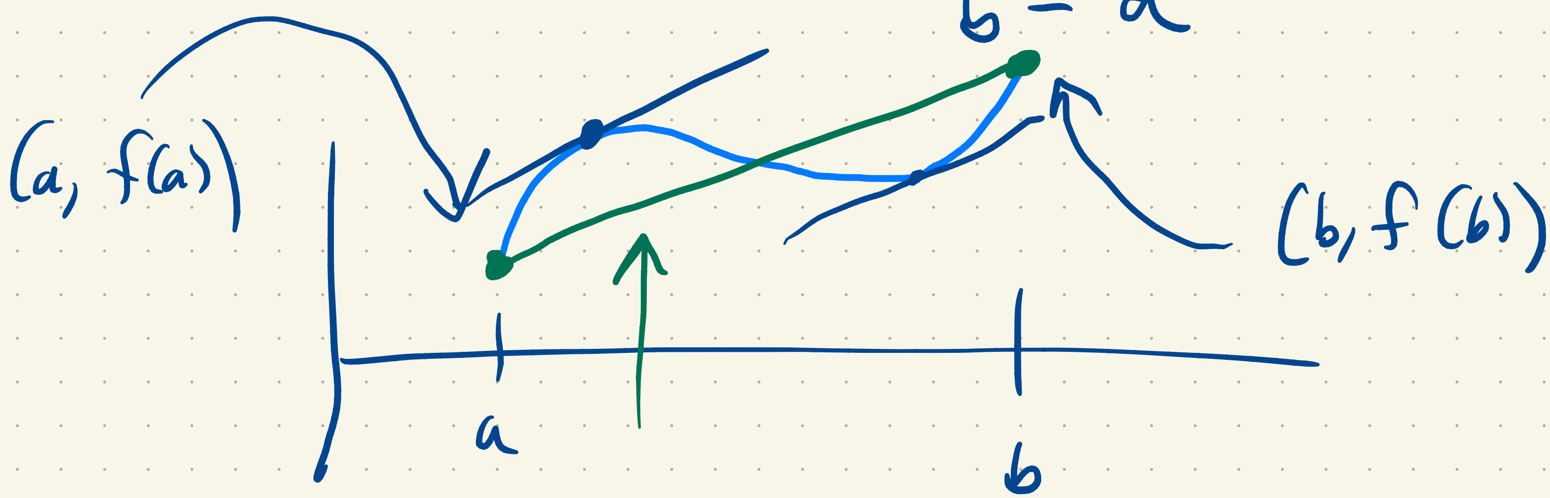


Pf: By the Extreme Value Theorem, the function achieves a maximum and a minimum value somewhere. If one of these is achieved at $c \in (a,b)$ then Fermat's theorem implies $f'(c) = 0$. If they are both achieved at the end points then, since $f(a) = f(b)$, the function is constant and $f'(c) = 0$ for all $c \in (a,b)$. $\square$

Cor (Mean Value Theorem)

Suppose $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



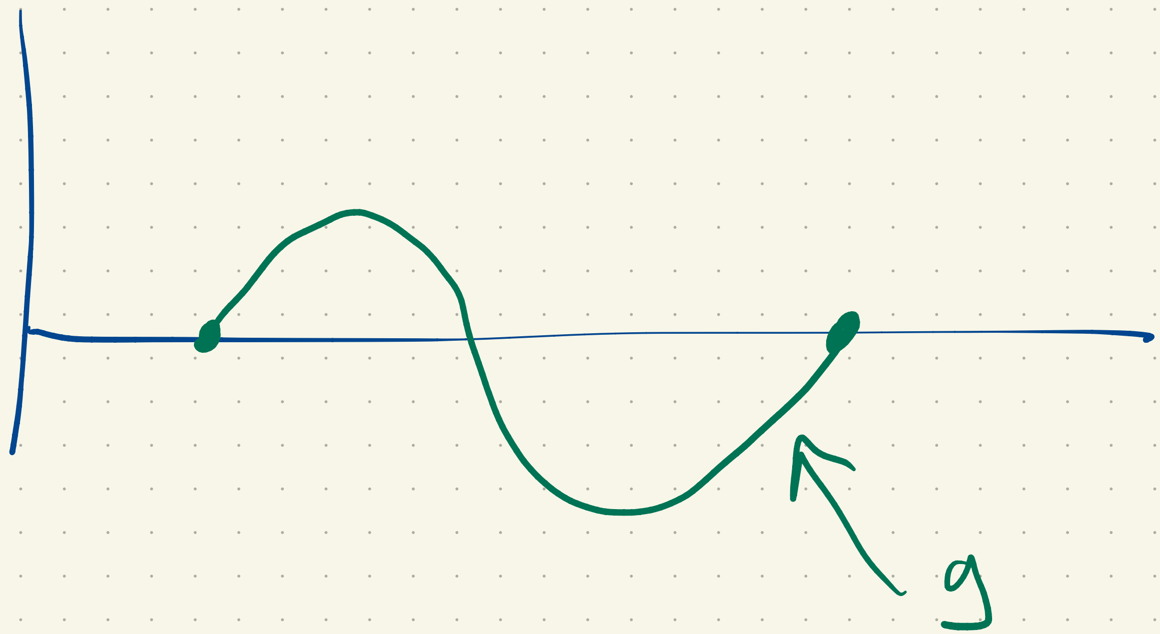
$$g(x) = f(x) - \left[ f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a} \right]$$

$$g(a) = f(a) - \left[ f(a)\frac{a-b}{a-b} + f(b)\frac{a-a}{b-a} \right]$$

$$= 0$$

$$g(b) = f(b) - \left[ f(a)\frac{b-b}{a-b} + f(b)\frac{b-a}{b-a} \right]$$

$$= 0$$




$c \in (a,b)$ $\qquad g'(c) = 0$

$$g(x) = f(x) - \left[ f(a)\,\frac{x-b}{a-b} + f(b)\,\frac{x-a}{b-a} \right]$$

$$g'(x) = f'(x) - \left[ -\frac{f(a)}{b-a} + \frac{f(b)}{b-a} \right]$$

$$= f'(x) - \left[ \frac{f(b) - f(a)}{b - a} \right]$$

$$g'(c) = 0 \iff f'(c) - \left[ \frac{f(b) - f(a)}{b - a} \right] = 0$$

$$\iff f'(c) = \frac{f(b) - f(a)}{b - a}$$

Suppose $f$ is continuous on $[a,b]$ and diff on $(a,b)$ and $f'(x)=0$ for all $x \in (a,b)$.

Then $f$ is constant.

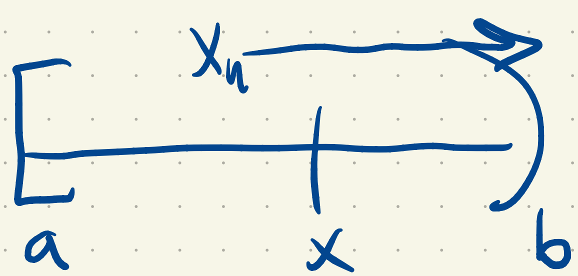
Mean value theorem: $\dfrac{f(x)-f(a)}{x-a} = f'(c) = 0$

for some $c$

where $a < c < x$.

$$\frac{f(x)-f(a)}{x-a} = 0$$

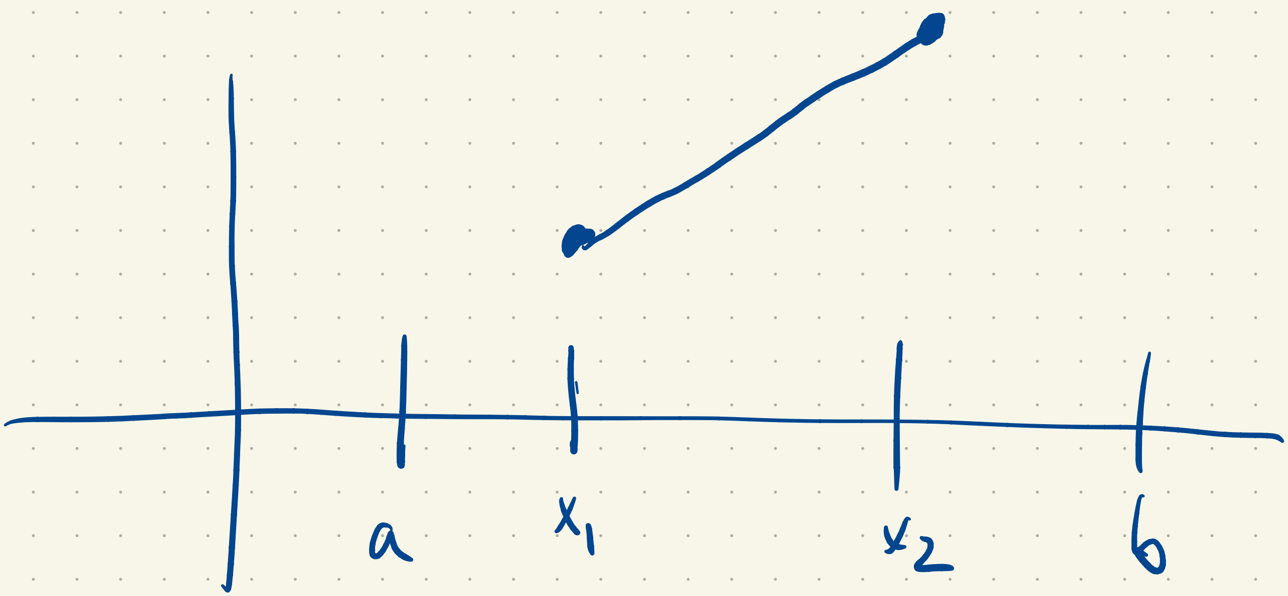
$$\Rightarrow f(x) = f(a)$$



$f : [a,b] \to \mathbb{R}$, cts.

$f$ is diff $(a,b)$

$f'(x) \geq 0$ for all $x \in (a,b)$.



$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) > 0$$

$$f(x_2) - f(x_1) > 0$$

$$f(x_2) > f(x_1)$$

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$f'(x) = g'(x)$ on $[a,b]$

$(f-g)'(x) = 0$ on $[a,b]$

$f - g = c \qquad f = g + c$