$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) = \lim_{x\to c}\beta(x) = \beta(c)$$

A function $f(x)$ is differentiable at c if and only if there exists a function μ that is continuous at c and such that

$$f(x) = f(c) + \overset{m}{\mu(x)}(x-c).$$

$$f(x) = f(c) + m(x-c)$$

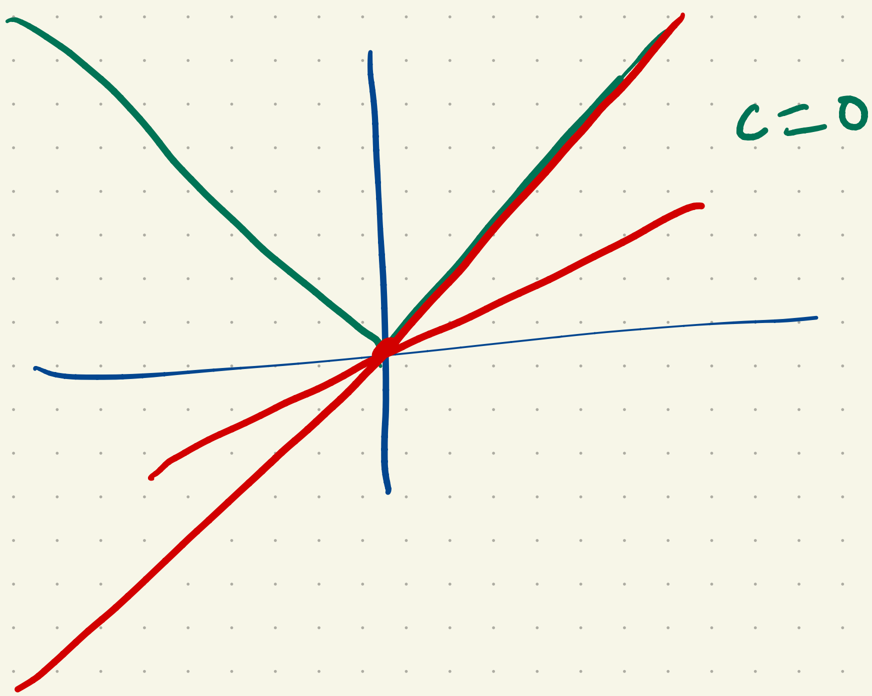
$$\mu(x) = m + \varepsilon(x)$$

$$m = \mu(c)$$

$\varepsilon(c) = 0$, ε is continuous at c

$$f(x) = f(c) + m(x-c) + \varepsilon(x)(x-c)$$

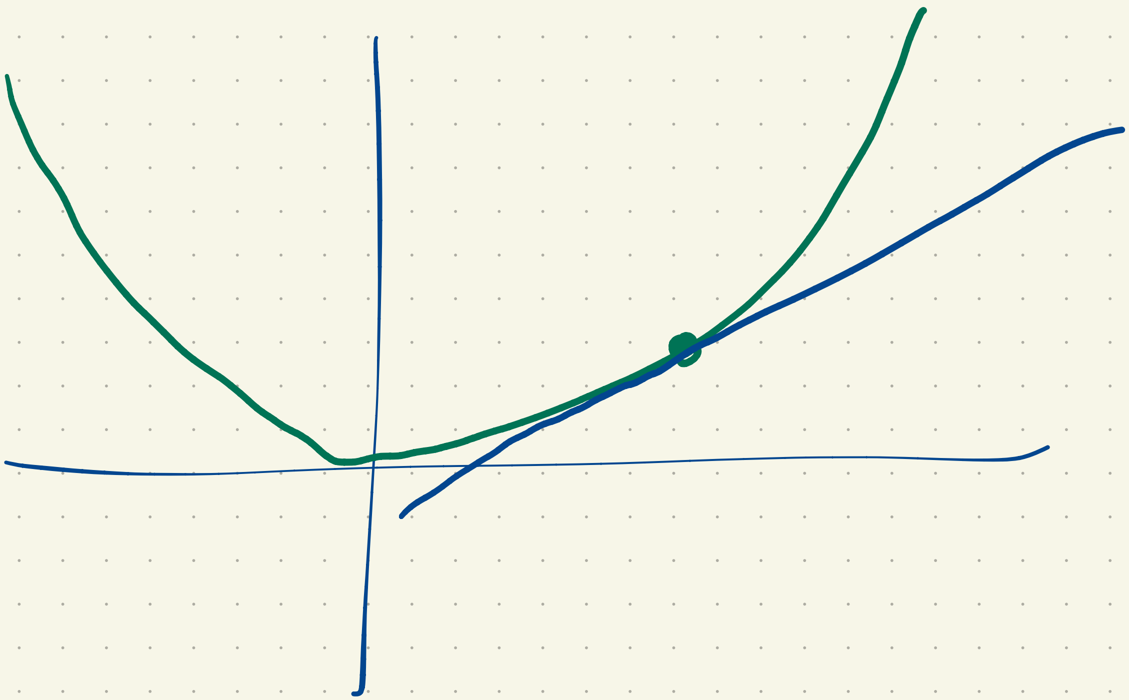
$\varepsilon \to 0$
as
$x \to c$



goes to 0

faster than

linearly.



Prop: If f is differentiable at c then it is continuous at c.

Sketch

$$f(x) = f(c) + \mu(x)(x-c)$$

$\hookrightarrow$ cts at c

$$f(x) = f(c) + \mu(x)(x-c)$$

$$g(x) = g(c) + \beta(x)(x-c)$$

$f+g \to$ diff at c

$$(f+g)'(c) = f'(c) + g'(c)$$

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= f(c) + \mu(x)(x-c) + g(c) + \beta(x)(x-c) \\ &= [f(c)+g(c)] + [\mu(x)+\beta(x)](x-c)\end{aligned}$$

$$= (f+g)(c) + \underbrace{[\mu(x)+\beta(x)]}(x-c)$$

μ is cts at c

$\mu(c) = f'(c)$

β is cts at c

$\beta(c) = g'(c)$

need to show this is continuous at c

in which case

$$(f+g)'(c) = [\mu(c)+\beta(c)]$$

$$= f'(c) + g'(c)$$

$$(f\cdot g)(x) = f(x)\,g(x)$$

$$(f\cdot g)'(c) = f(c)\,g'(c) + f'(c)\,g(c)$$

$$f(x) = f(c) + \mu(x)(x-c)$$
$$g(x) = g(c) + \beta(x)(x-c)$$

$$(f\cdot g)(x) = f(x)\cdot g(x)$$
$$= \big(f(c) + \mu(x)(x-c)\big)\big(g(c) + \beta(x)(x-c)\big)$$

$$= f(c)g(c) + \mu(x)(x-c)g(c) + \beta(x)(x-c)f(c) + \mu(x)\beta(x)(x-c)^2$$

$$(f\cdot g)(x) = (f\cdot g)(c) + [\mu(x)g(c) + \beta(x)f(c) + \mu(x)\beta(x)(x-c)](x-c)$$

show this part is continuous at c
in which case $(f\cdot g)'(c)$ is the value at c.

$$\underbrace{\mu(c)}_{f'(c)}\, g(c) + \underbrace{\beta(c)}_{g'(c)}\, f(c) + \cancelto{0}{\mu(c)\beta(c)(c-c)}$$

$$\underline{f'(c)\,g(c) + f(c)\,g'(c)}$$

$f : A \to \mathbb{R}$

$g : B \to \mathbb{R}$

$g \circ f$ $\qquad$ $f(A) \subseteq B$

c, limit point of

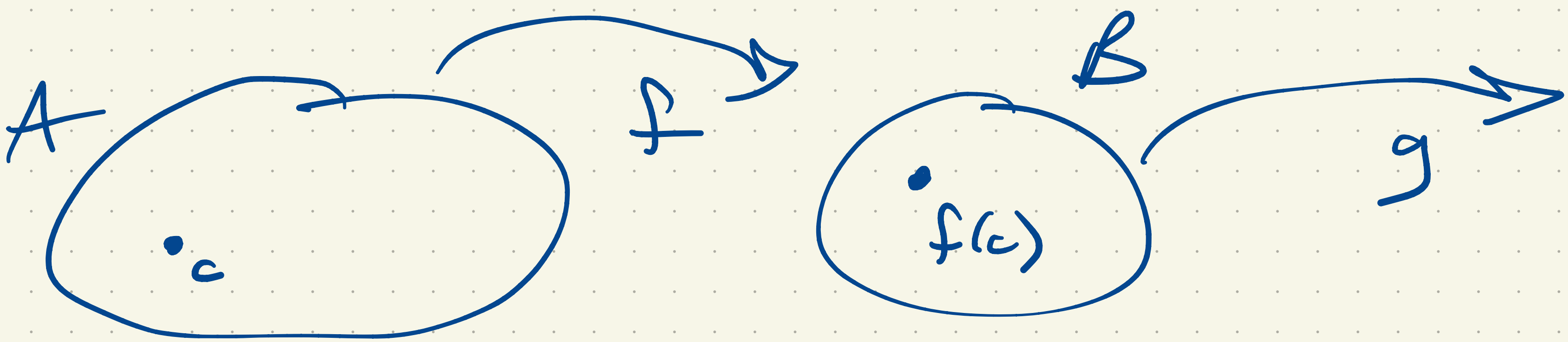
$c \in A$

f is diff at c

$f(c)$ is a limit point of B

$f(c) \in B$

g is diff at $f(c)$



$g \circ f$ is diff at c

$$(g \circ f)'(c) = g'(f(c))\, f'(c)$$

$$f(x) = f(c) + \mu(x)(x-c)$$

$$g(z) = g(f(c)) + \beta(z)(z - f(c))$$

$$g(f(x)) = g(f(c)) + \beta(f(x))(f(x) - f(c))$$

$$= g(f(c)) + \underbrace{\beta(f(x))\mu(x)}(x-c)$$

β is cont. at $f(c)$

eval at c

$$\beta(f(c)) \cdot \mu(c)$$

$$\boxed{g'(f(c))\, f'(c)}$$

Since f is diff at c it is cts at c.

Then using standard continuity results

$\beta(f(x))\,\mu(x)$ is continuous at c.

Lemma : Consider $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$$f(x) = \frac{1}{x}.$$

Then for all $c \in \mathbb{R}$, $f'(c) = -\frac{1}{c^2}$.

Pf: (Easy, from def).

$f : A \to \mathbb{R}$ $\quad f(x) \neq 0 \ \forall x \in A.$

f is diff at c.

$$g(x) = \frac{1}{f(x)} \qquad \frac{1}{z}$$

By the chain rule, $g'(c) = \frac{-1}{(f(c))^2} \cdot f'(c)$

$f(x)$, as before.

$h(x)$, diff at c.

$$\left(\frac{h}{f}\right)'(c) = \left(h \cdot \frac{1}{f}\right)'(c)$$

$$= h'(c)\frac{1}{f(c)} + h(c) \cdot \left(\frac{-1}{f(c)^2}\right) f'(c)$$

$$= \frac{h'(c)f(c)}{(f(c))^2} - \frac{f'(c)h(c)}{(f(c))^2}$$

$$= \frac{f(c)h'(c) - f'(c)h(c)}{(f(c))^2}.$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\boxed{\frac{d}{dx} x = 1}$$

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx} x x^n$$

$$\boxed{\begin{array}{l} \text{Lemma: Consider } f: \mathbb{R}\setminus\{0\} \to \mathbb{R} \\ \qquad f(x) = \frac{1}{x}. \\ \text{Then for all } c \in \mathbb{R}, \quad f'(c) = -\frac{1}{c^2}. \\ \text{Pf: (Easy, from def).} \end{array}}$$