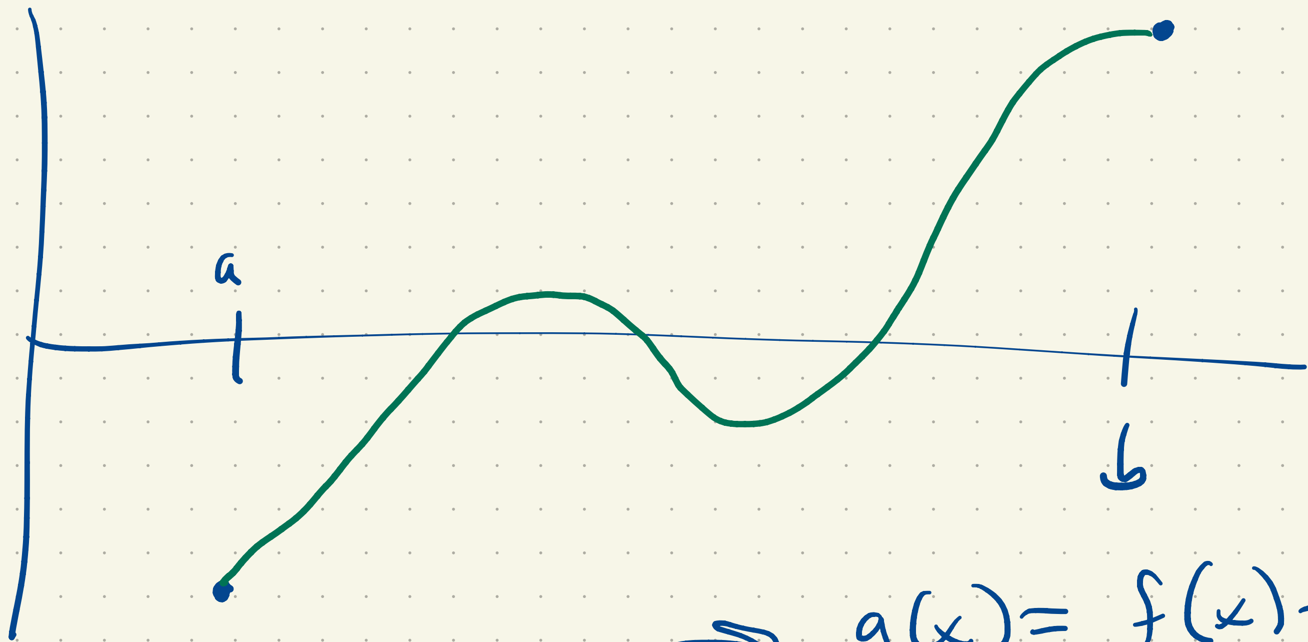
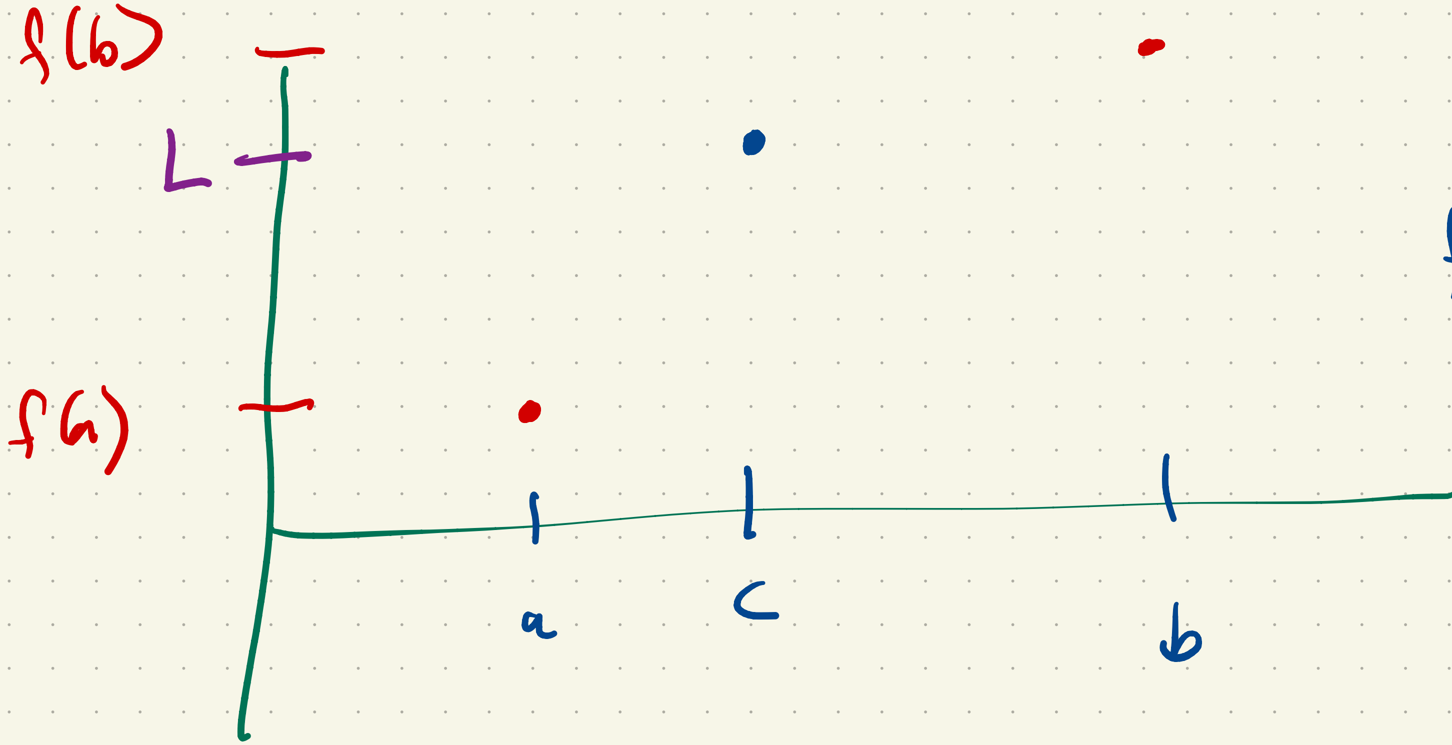



$g(x) = f(x) - L$

$g(a) = f(a) - L$

IVT: Suppose $f : [a,b] \to \mathbb{R}$ is continuous and $L \in \mathbb{R}$ with either $f(a) < L < f(b)$ or $f(a) > L > f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = L$.




$$f(a) < L$$

$$f(a) - L < 0$$

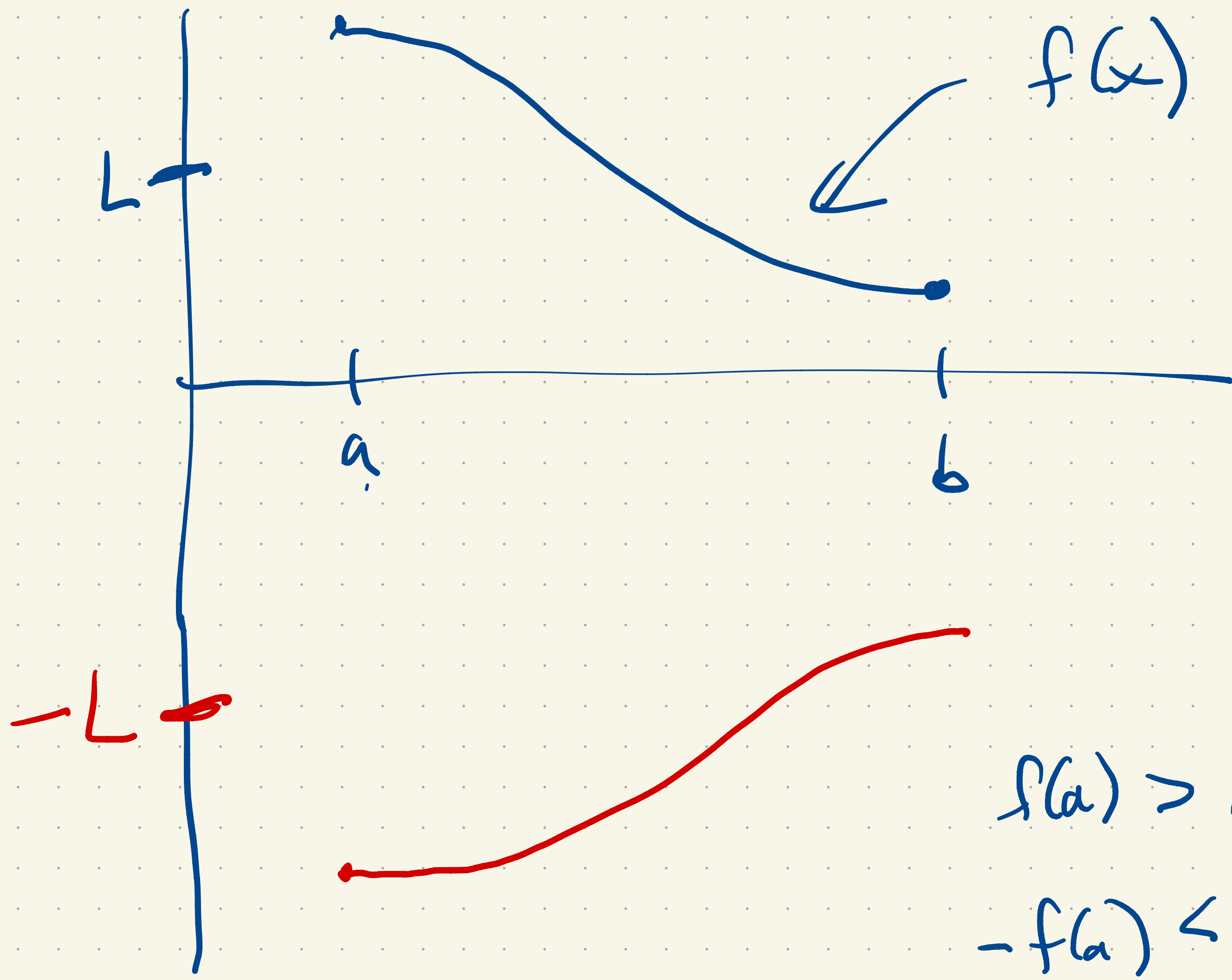
$$L < f(b)$$

$$0 < f(b) - L$$

$$0 < g(b)$$

$$g(c) = 0$$

$$f(c) - L = 0 \Rightarrow f(c) = L$$




$g(x) = -f(x)$

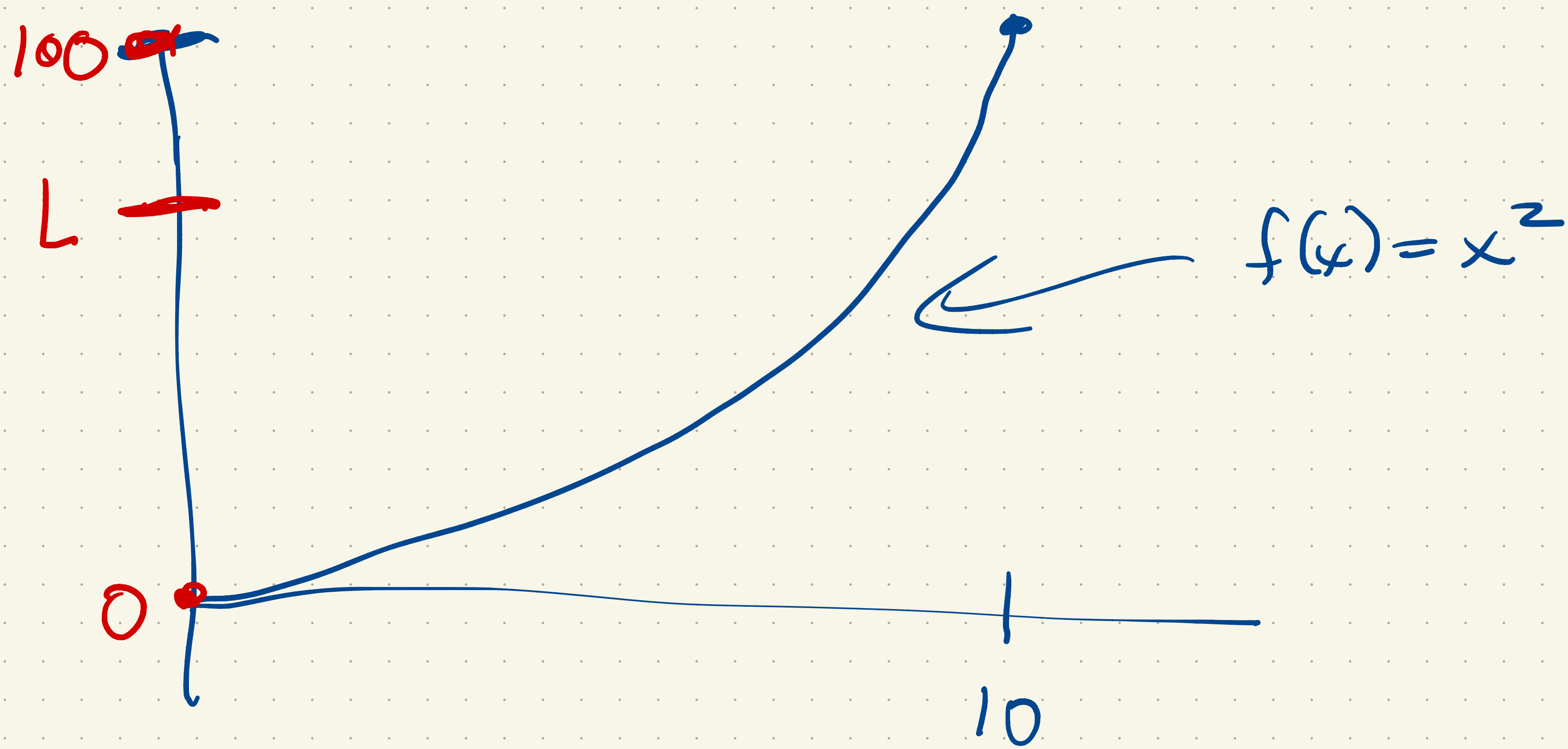
$f(a) > L > f(b)$

$-f(a) < -L < -f(b)$

$g(a) < -L < g(b)$

$g(c) = -L$

$f(c) = L \longleftarrow -f(c) = -L$




$f(0) = 0 \qquad f(10) = 100$

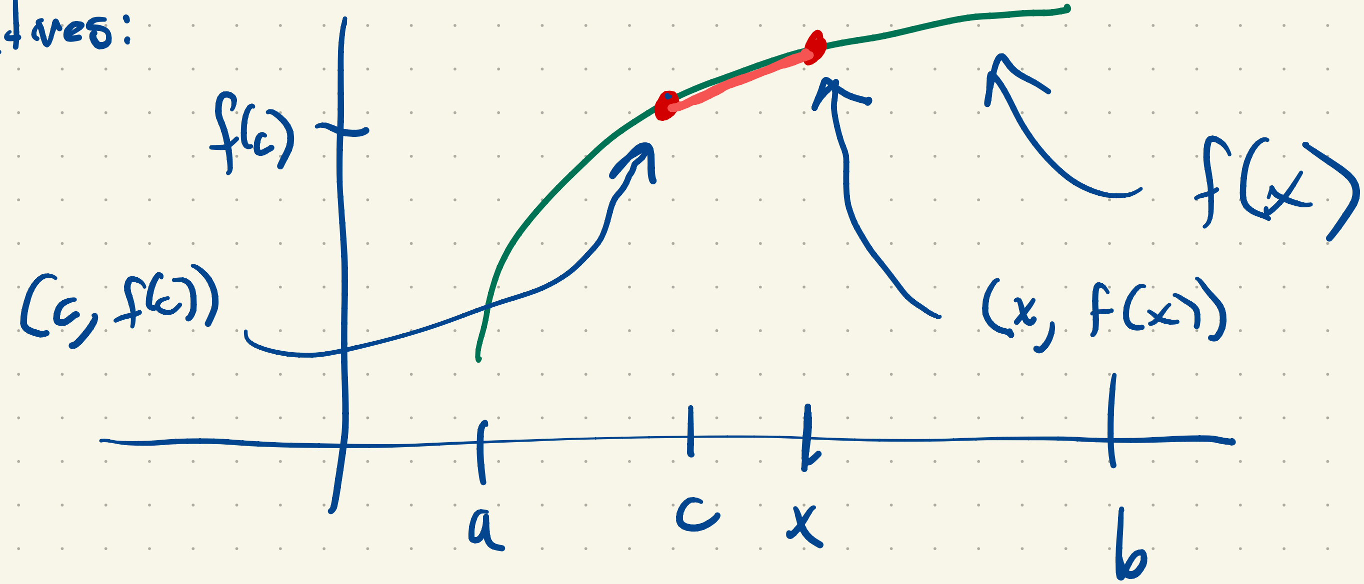
There exists $c$ in $(0, 10)$

such that $c^2 = L$.

2

$c^2 = 2$

Derivatives:



$$f'(c) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

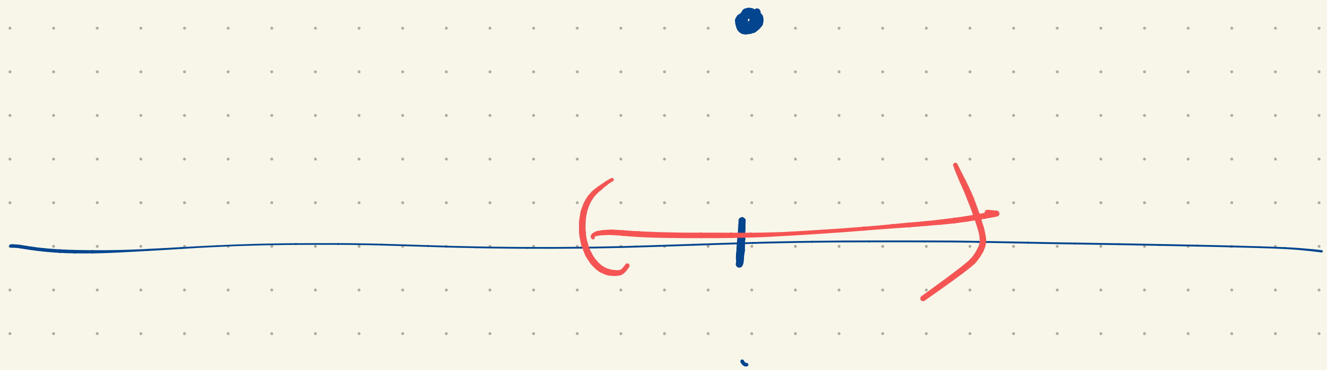
$\frac{\text{rise}}{\text{run}}$

$$\frac{f(x)-f(c)}{x-c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Def: Suppose $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and that $c \xrightarrow{} \in A$ is a limit point of $A$.

We say $f$ is differentiable at $c$ if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists, in which case we write $f'(c)$ for the limiting value.



Suppose $f$ is differentiable at $c$.

Define
$$\mu(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

Observe:
$$\lim_{x \to c} \mu(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$$

$$= \mu(c)$$

$$\boxed{\lim_{x \to c} \mu(x) = \mu(c)}$$

The function $\mu$ is continuous at $c$.

$$\mu(x) = \frac{f(x) - f(c)}{x - c} \qquad (x \neq c)$$

$$f(x) = f(c) + \mu(x)(x - c) \qquad (x \neq c)$$

$$(x = c)$$

If $f$ is differentiable at $c$

then there exists a ~~continuous~~ function

$\mu$ that is continuous at $c$ such that

$$f(x) = f(c) + \mu(x)(x-c)$$

and $\mu(c) = f'(c)$.

$$\mu(c) = m$$

If $x$ is near $c$ $\quad \mu(x) \simeq m$

$$f(x) \simeq f(c) + m(x-c)$$

There is a converse.

Suppose $\beta$ is continuous at $c$

and $f(x) = y_0 + \beta(x)(x-c)$

I claim $f$ is differentiable at $c$

and $\beta(c) = f'(c)$.

$$f(c) = y_0 + 0 = y_0$$

$$\frac{f(x)-f(c)}{x-c} = \frac{f(x)-y_0}{x-c} = \beta(x)$$

$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) = \lim_{x\to c}\beta(x) = \beta(c)$$

---

A function $f(x)$ is differentiable at $c$ if and only if there exists a function $\mu$ that is continuous at $c$ and such that

$$f(x) = f(c) + \overset{m}{\mu}(x)(x-c).$$