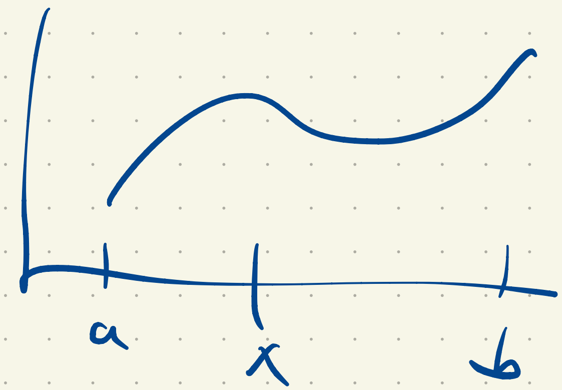
Last class:

FTC



1) $F'(x) = f(x)$ on $[a,b]$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

f Riemann int on $[a,b]$

2)

$$\frac{d}{dx} \int_a^x f(\text{☺})\,d\text{☺} = f(x)$$

f is continuos

1) If you want to compute a definite integral, find an antiderivative

2) If you want an antiderivative, you can make one using definite integrals

$$F(x) = \int_a^x f(\text{☺})\,d\text{☺} \qquad F'(x) = f(x)$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Please read text concerning this.

Lemma: Suppose f is Riemann integrable on $[a,b]$. Then there is a sequence of partitions P_n such that

$$U(f, P_n) \to \int_a^b f \quad \text{and} \quad L(f, P_n) \to \int_a^b f.$$

Conversely if f is bounded on $[a,b]$ and there exist partitions P_n such that

$U(f, P_n) \to L$ and $L(f, P_n) \to L$

for some L then f is Riemann integrable on $[a,b]$ and $\int_a^b f = L$.

Sequences of functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{0.1} = \sum_{k=0}^{\infty} \frac{(0.1)^k}{k!}$$

a_k

$$e^{0.1} = \lim_{n\to\infty} s_n$$

$$s_n = \sum_{k=0}^{n} \frac{(0.1)^k}{k!}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$P_4(x)$

$P_4'(x)$

$$\lim_{n \to \infty} P_n = \exp$$

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$P_4'(x) = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}_{\to P_3(x)}$$

$$\exp(x) = e^x$$

$$\exp'(x) = e^x = \exp(x)$$

$$\exp(x) = \lim_{n \to \infty} p_n(x)$$

$$\frac{d}{dx} \exp(x) = \frac{d}{dx} \lim_{n \to \infty} p_n(x) = \lim_{n \to \infty} \frac{d}{dx} p_n(x)$$

$$= \lim_{n \to \infty} p_{n-1}(x)$$

$$\lim_{h \to 0}$$

$$= \exp(x)$$

$$\lim_{h\to 0}\ \lim_{n\to\infty} \quad \text{vs} \quad \lim_{n\to\infty}\ \lim_{h\to 0}$$

Def: Let $f_n : A \to \mathbb{R}$ be a sequence of functions.

Suppose $f : A \to \mathbb{R}$.

We say $f_n \to f$ <u>pointwise</u> if for all $x \in A$, $f_n(x) \to f(x)$.

e.g. $f_n(x) = \sqrt{\frac{1}{n} + x^2}$

$f_n(0) = \sqrt{\frac{1}{n}} \to 0$

$f_n(3) = \sqrt{\frac{1}{n} + 9} \to 3$

$f_n(-2) \longrightarrow 2$

$f_n \longrightarrow \text{abs}$ pointwise

$$f_n(x) = \sqrt{\frac{1}{n} + x^2}$$

$$f_n'(x) = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{n} + x^2}} \cdot 2x$$

$$= \frac{x}{\sqrt{\frac{1}{n} + x^2}}$$

$$f_n' \longrightarrow \text{abs}'$$

$$\frac{x}{|x|} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$$f_n'(0) = 0$$

e.g. $f_n(x) = x^n$ on $[0,1]$

$f_n(0) = 0^n \to 0$

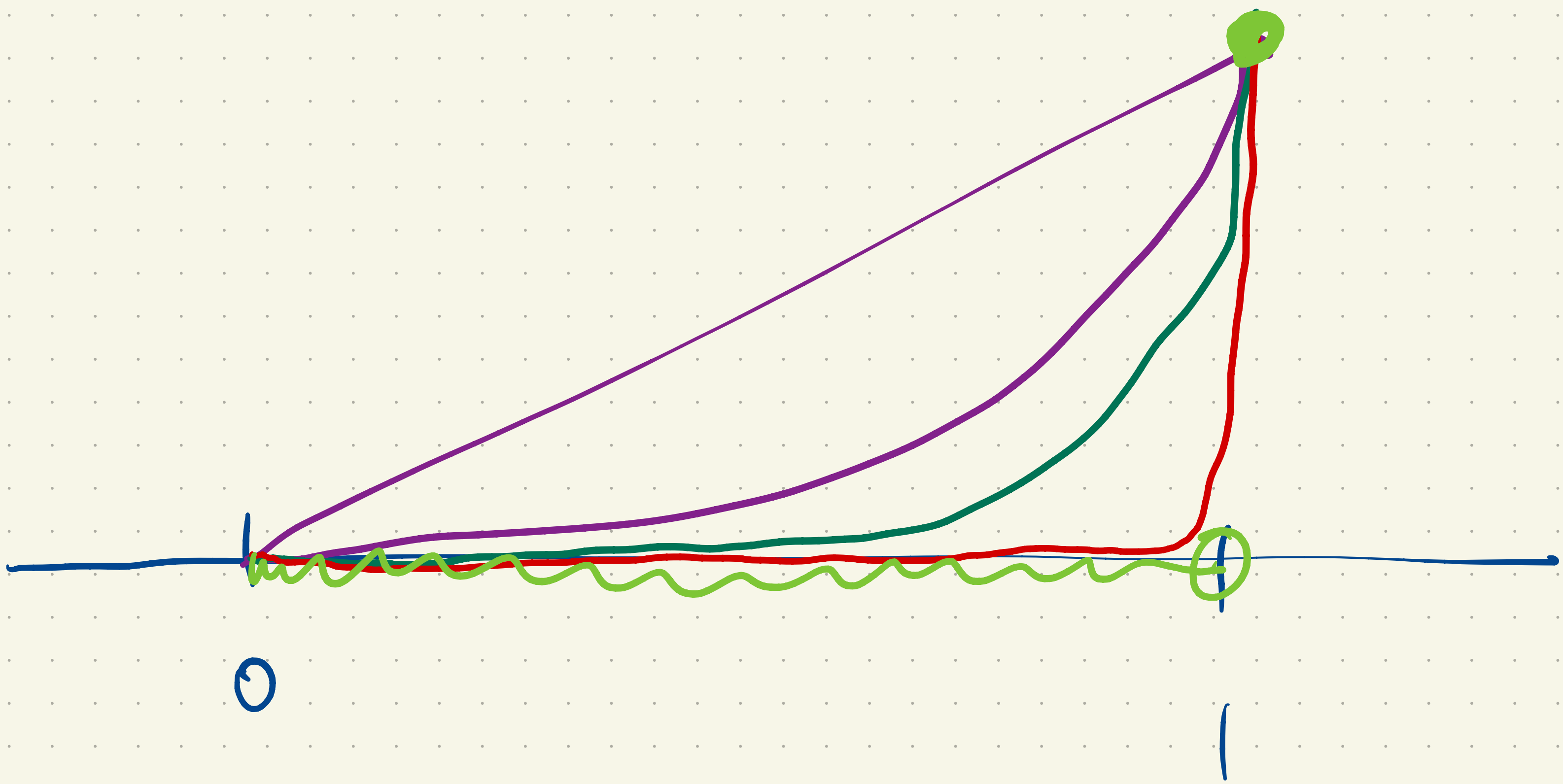
$f_n(1/2) = \left(\frac{1}{2}\right)^n \to 0$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

$f_n(x) \longrightarrow g(x)$ pointwise

$$g(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

$f_n(x) = x^n$



The pointwise limit of continuous functions need not be continuous.

The pointwise limit of diff functions need not be diff.

Suppose $f_n \to f$ pointwise.

Each f_n is continuous. How come f need not be continuous?

f_n is cts at c.

want f is cts at c

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| \qquad |x-c| < \delta$$

$$+ |f_n(x) - f_n(c)|$$

$$+ \underbrace{|f_n(c) - f(c)|}$$

Given $\varepsilon > 0$ we can find N so

$$\left| f_N(c) - f(c) \right| < \frac{\varepsilon}{3}.$$

Since f_N is continuous at c

we can find a $\delta > 0$ so if $|x-c| < \delta$

then $$\left| f_N(x) - f_N(c) \right| < \frac{\varepsilon}{3}.$$

If $|x-c|<\delta$

$$|f(x)-f(c)| < |f(x)-f_N(x)| + \frac{2\varepsilon}{3}$$

we have no guarantee that for all x with $|x-c|<\delta$, $|f(x)-f_N(x)|<\frac{\varepsilon}{3}$

Def: A sequence of functions $f_n \to f$ <u>uniformly</u> if for all $\varepsilon > 0$ there exists N so that if $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in A$. $\quad (f_n, f : A \to \mathbb{R})$