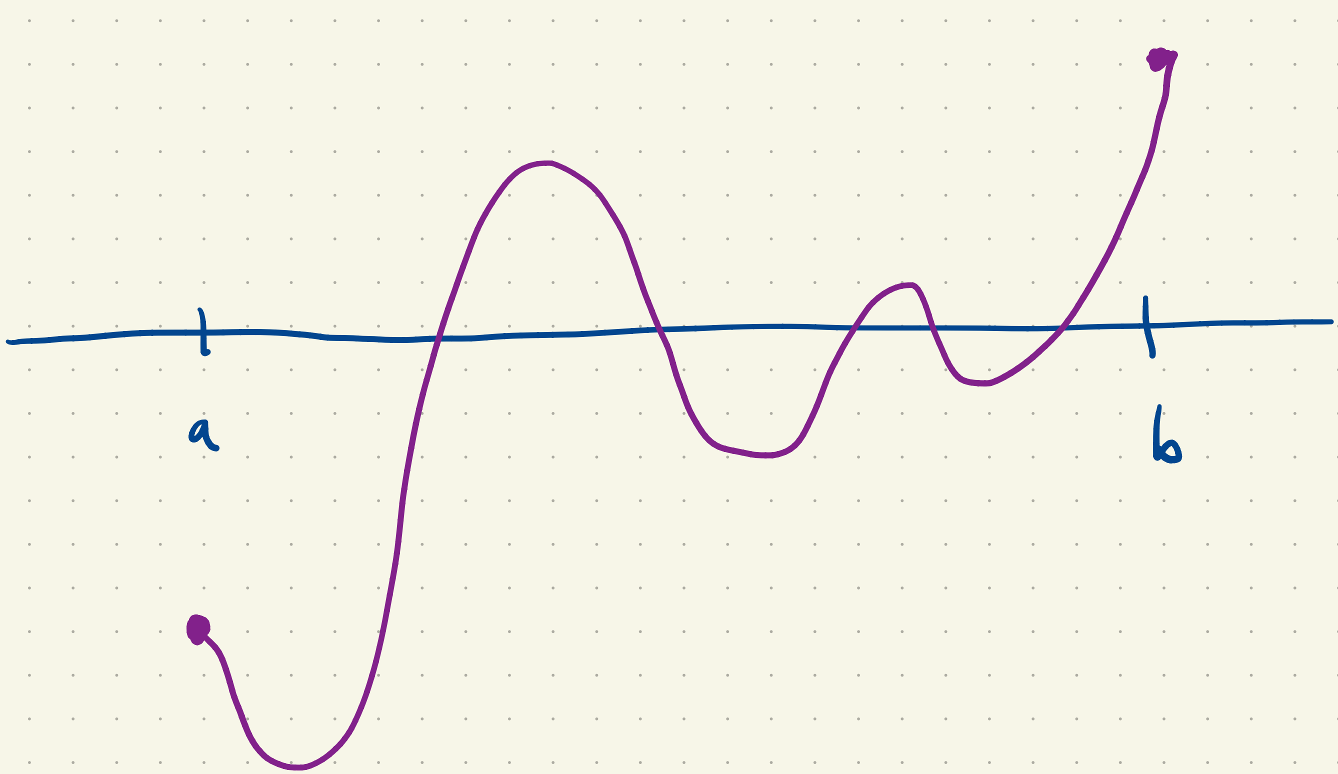
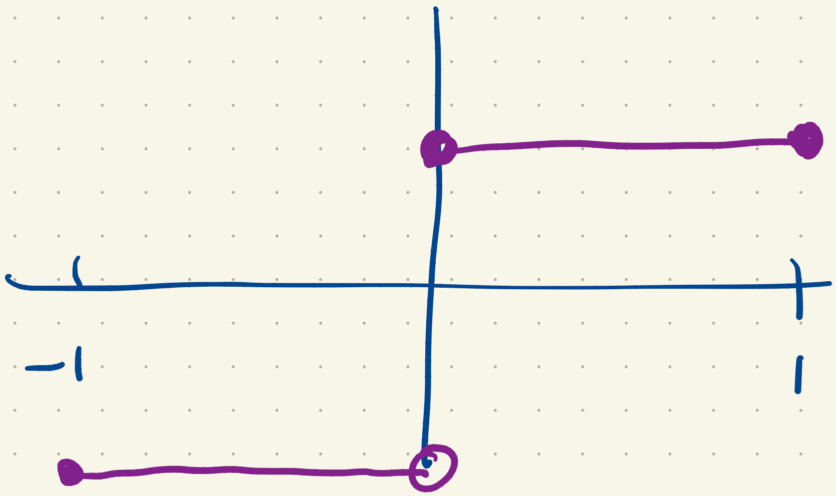
Basic Version (IVT)



- $f : [a,b] \to \mathbb{R}$
- $f(a) < 0$
- $f(b) > 0$
- f is continuous

There is an $x \in (a,b)$ where $f(x) = 0$.

Continuity is needed: $f(x) = \begin{cases} -1 & -1 \leq x < 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$



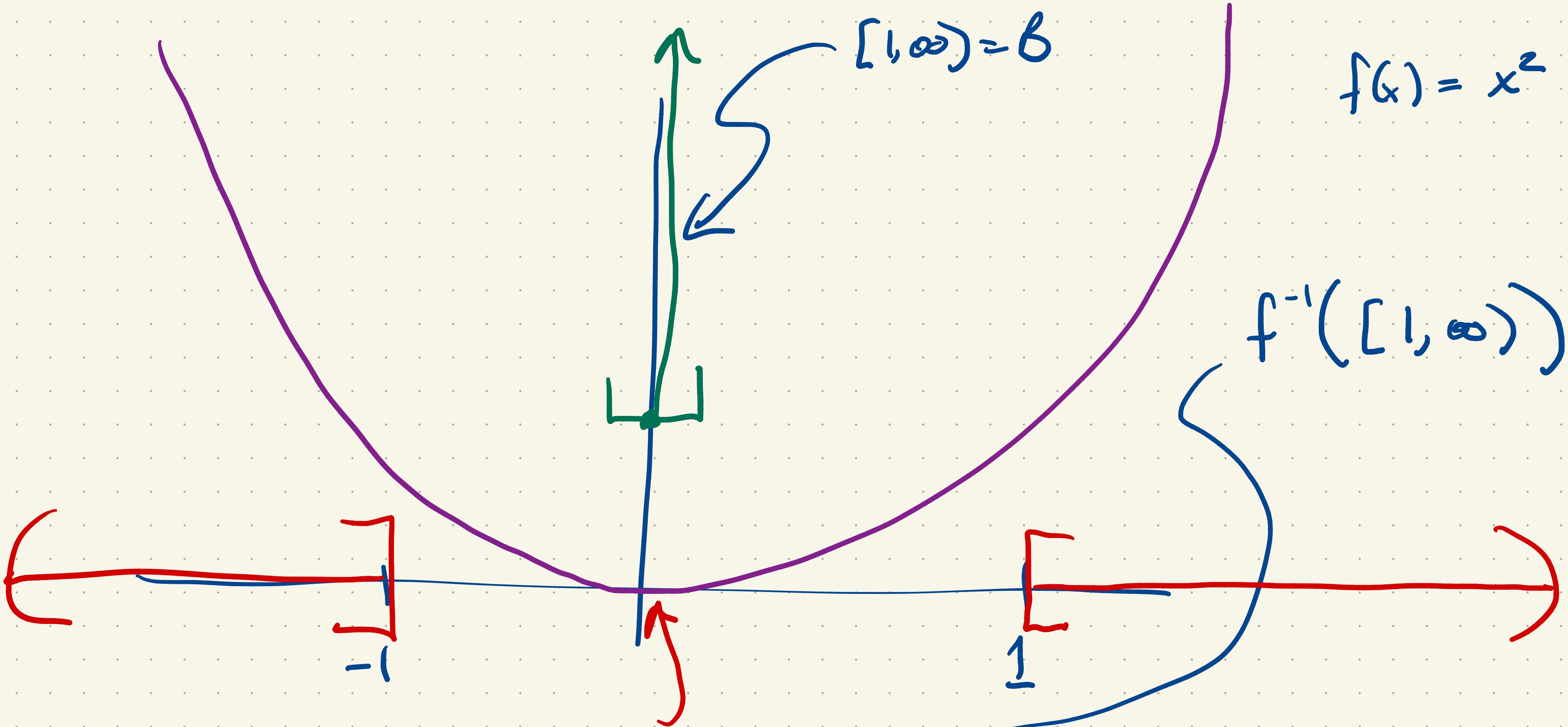
Tool: preimage of a set.

$$f : A \to \mathbb{R}$$

$$B \subseteq \mathbb{R}$$

$$f^{-1}(B) = \{ a \in A : f(a) \in B \}$$

$f(x) = x^2$



$\{x \in \mathbb{R} : f(x) \in [1,\infty)\}$

$0 \in f^{-1}([1,\infty))$?

No

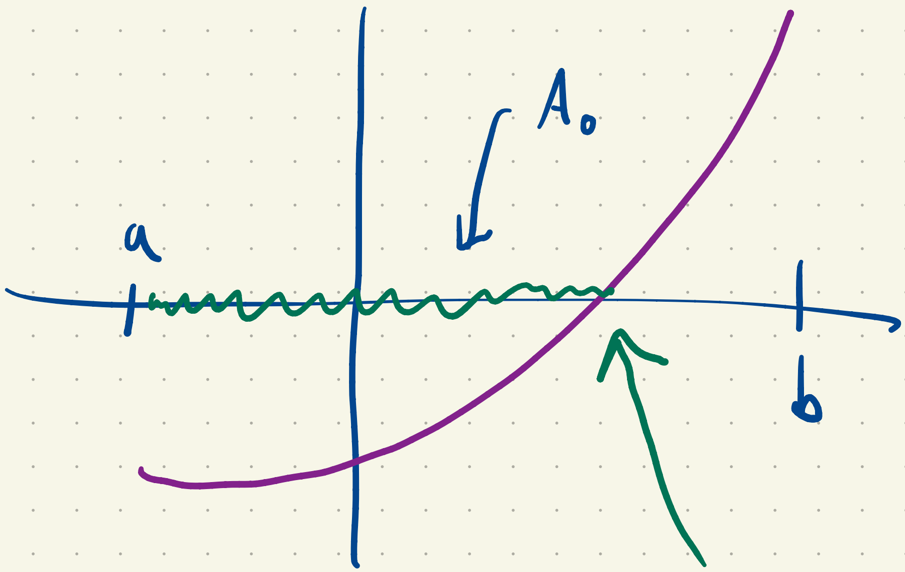
$f(0) = 0 \notin [1,\infty)$

$$f : \underbrace{[a,b]}_{A} \to \mathbb{R}$$

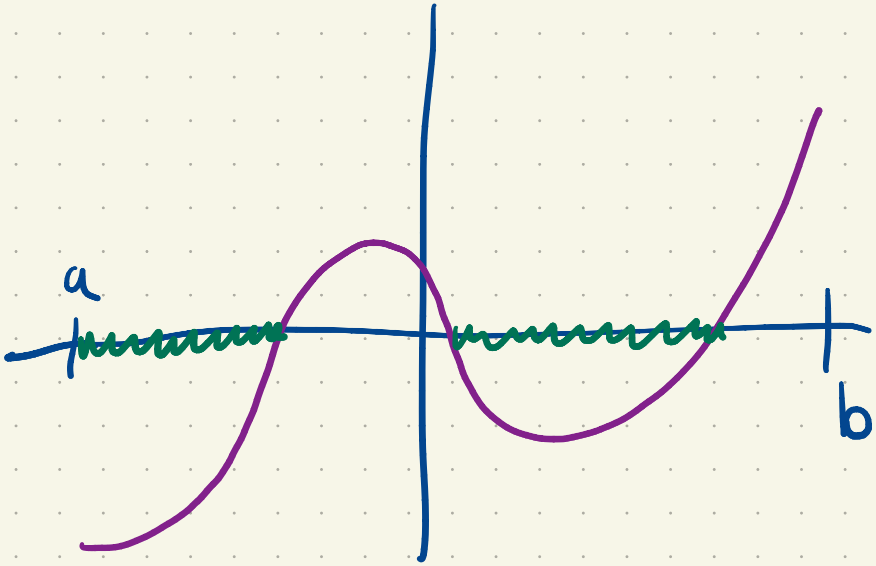
$$B = (-\infty, 0]$$

$$A_0 = f^{-1}\big((-\infty,0]\big) = \{x \in [a,b] : f(x) \in (-\infty,0]\}$$

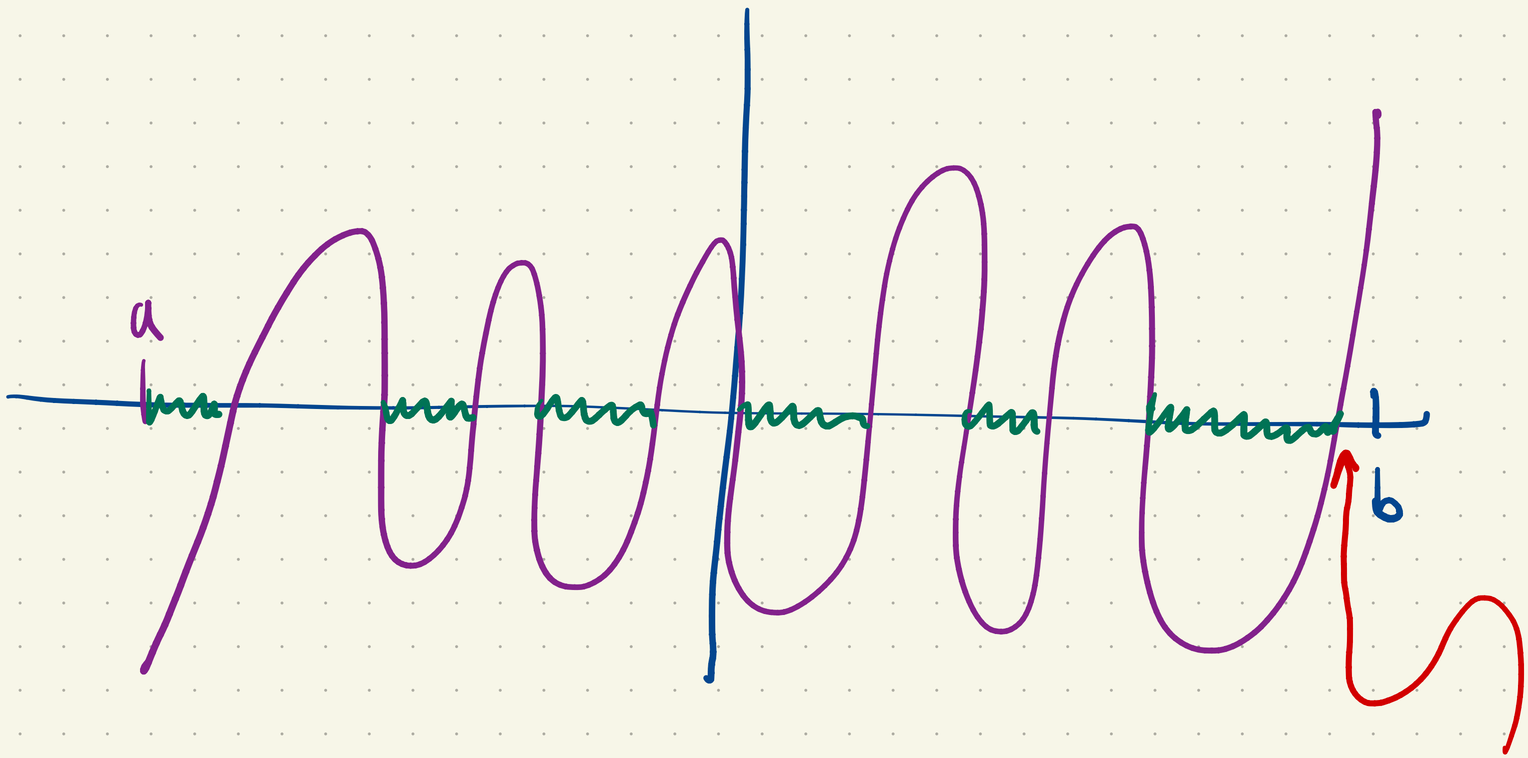
$$= \{x \in [a,b] : f(x) \leq 0\}$$



$$A_0 = f^{-1}((-\infty, 0])$$
$$= \{x \in [a,b] ; \ f(x) \le 0\}$$



$$f(x) = 0$$



We hope that $x = \sup(A_0)$

has $f(x) = 0$.

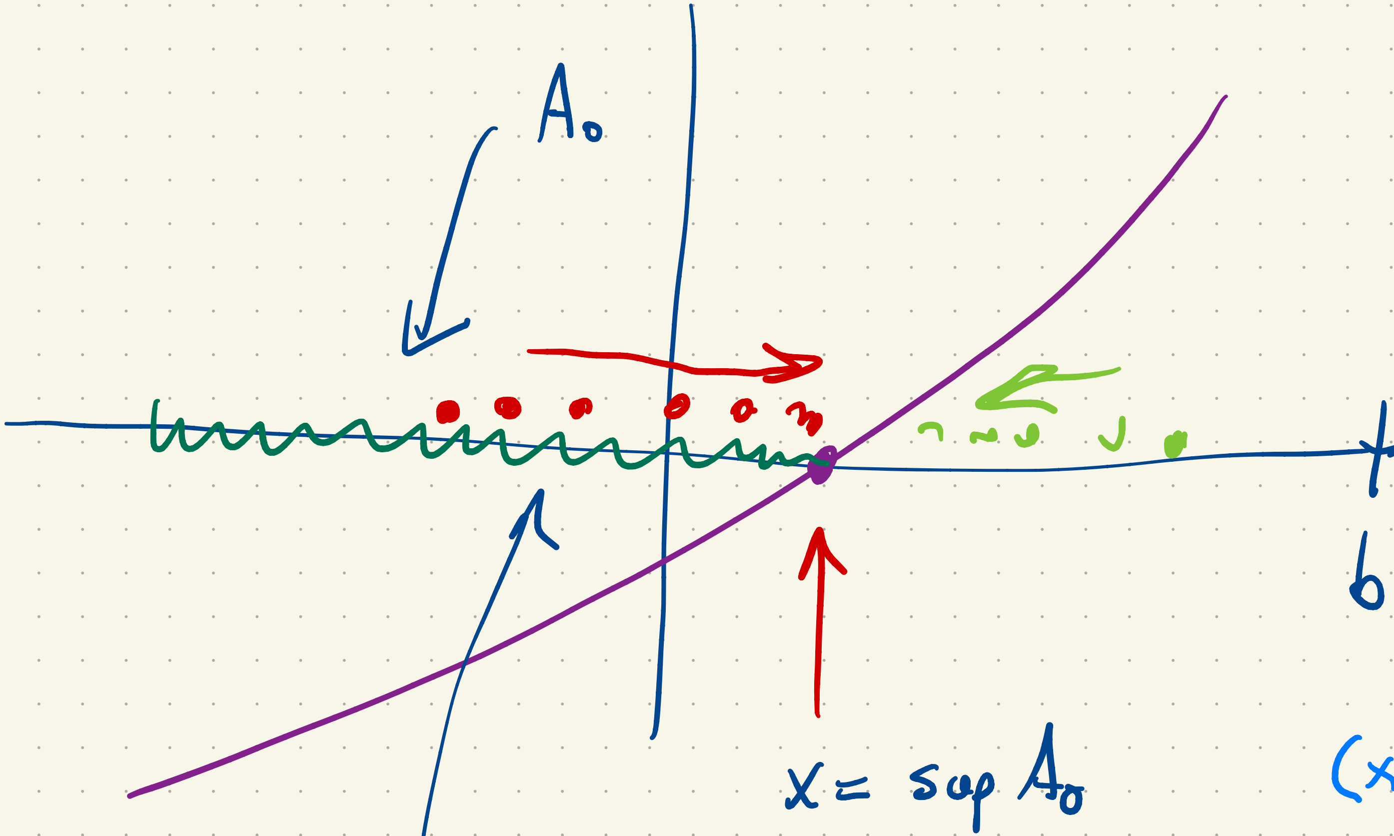
is A_0 bounded above? ✓

is $A_0 \neq \emptyset$?

nope! $f(a) < 0$ so $a \in A_0$

$A_0 \subseteq [a, b]$

$\hookrightarrow$ b is an upper bound.



$f(x) \leq 0$

$f(x) \geq 0?$

$x + \frac{1}{n} \in (x, b]$

$x + \frac{b-x}{n}$

$(x, b] \subseteq A_0^c$

$(0, 1]$ $\frac{1}{n}$

$(7, 8]$ $7 + \frac{1}{n}$

$a_n \in A_0$

$a_n \to x \Rightarrow f(a_n) \to f(x)$

$f(a_n) \leq 0 \Rightarrow f(x) \leq 0.$

$w_n \leq 0 \qquad w_n \to w \implies w \leq 0$

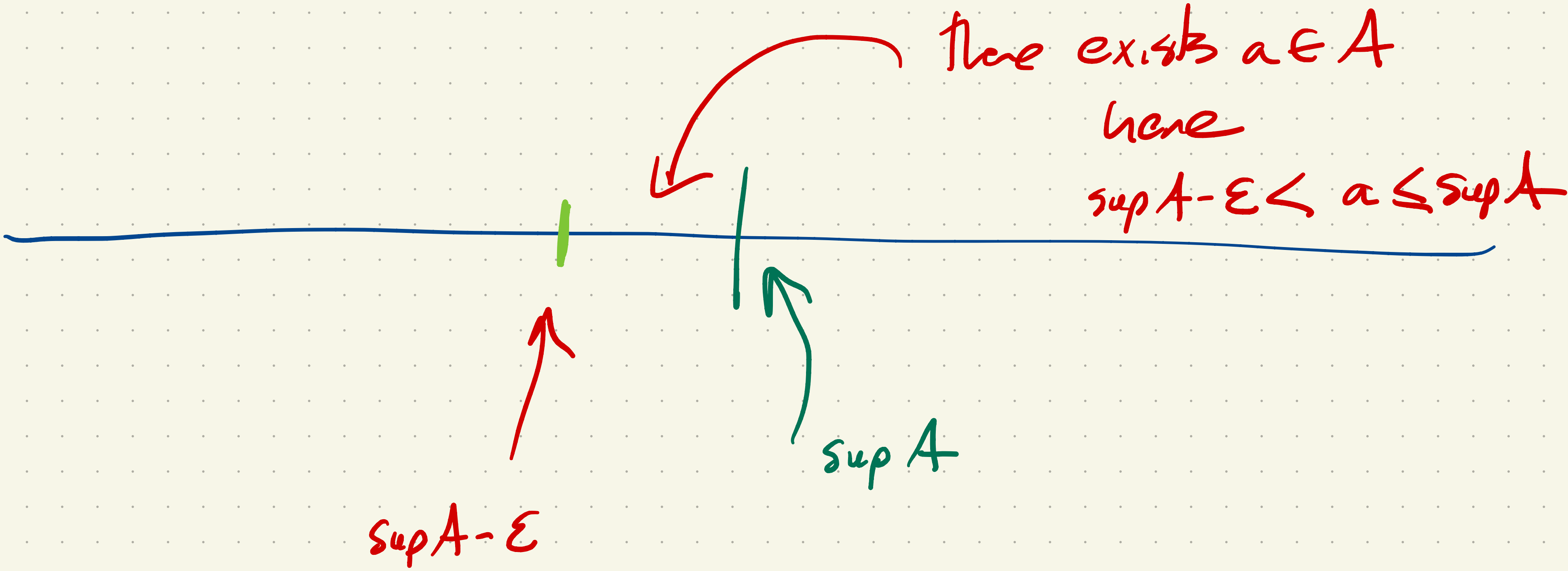
Limit order theorem

Lemma: Suppose $A \subseteq \mathbb{R}$ is nonempty and bounded above and hence admits a supremum $x = \sup(A)$. Then there exists a sequence (a_n) in A with $a_n \to x$.

$\varepsilon = \frac{1}{n}$

Pf: By Lemma 1.3.8 for each $n \in \mathbb{N}$ there exists $a_n \in A$ such that $x - \frac{1}{n} < a_n \leq x$.

Then, by the squeeze theorem, $a_n \to x$.



$\varepsilon > 0$

$\underbrace{\sup A - \varepsilon} < \sup A$

$\Rightarrow \sup A - \varepsilon$ is not an upper bound

$\sup A - \varepsilon$ is not an upper bound of A

$\Rightarrow$ there exists $a \in A$ s.t.

$\sup A - \varepsilon < a \leq \sup A$

Pf: (IVT, basic version)

Suppose $f : [a,b] \to \mathbb{R}$ is continuous and

$f(a) < 0 < f(b)$.

Let $A_0 = f^{-1}((-\infty, 0])$. Since $f(a) < 0$,

$a \in A_0$ and $A_0 \neq \emptyset$. Since $A_0 \subseteq [a,b]$, it is bounded above. Thus A_0 admits a supremum x and we claim $f(x) = 0$.

First, let (a_n) be a sequence in A_0 converging to x; such a sequence exists by the previous lemma. Observe that $f(a_n) \leq 0$ for all n. By continuity $f(a_n) \to f(x)$ so $f(x) \leq 0$ also.

Since $f(b) > 0$, $x \neq b$. Thus the

interval $(x, b]$ is nonempty. Since x is the supremum of A_0 and since each $z \in (x, b]$ satisfies $z > x$, it follows that $z \notin A_0$ so $f(z) > 0$. Let (b_n) be a sequence in $(x, b]$ converging to x (for example, $x + \frac{b-x}{n} = b_n$ will work). Then $f(b_n) > 0$ for each n and $b_n \to x$ so by continuity

$f(b_n) \to f(x)$ and $f(x) \geq 0.$

Hence $f(x) = 0.$ $\square$