Properties of Riemann Integral

a) $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

b) $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$f(x) \leq g(x)$ on $[a,b]$

$M_k^f \qquad M_k^g$

c) $\int_a^b f(x) \leq \int_a^b g(x)$

$x = 3 \qquad y = -3$

$|x+y| \leq |x| + |y|$

$$|x_1 + x_2 + x_3 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

d) $$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

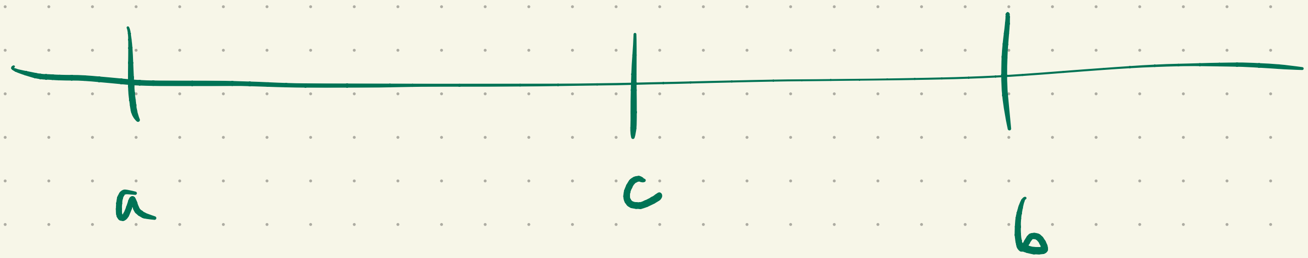
Suppose $\int_a^b f(x)\,dx \geq 0$

$$f(x) \leq |f(x)|$$

$$\left| \int_a^b f(x)\,dx \right| = \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

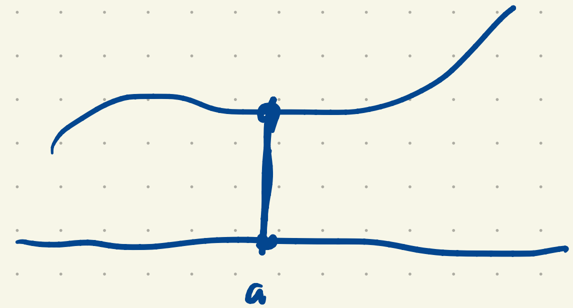
Exercise: prove the remaining case $\int_a^b f(x)\,dx < 0$

by considering $-f(x)$ $\qquad -f(x) \le |f(x)|$



$$\int_a^b f = \int_a^c f + \int_c^b f \qquad a < c < b$$

We'll define $\int_a^a f = 0$



If $a<b$

$$\int_b^a f = -\int_a^b f$$

Δx_k

Exercise: If f is Riemann int on I and if $a, b, c \in I$ then

$$\int_a^c f + \int_c^b f = \int_a^b f$$

regardless of the choice of a, b, c.

$c < a < b$

$$\int_c^b f = \int_a^b f - \int_a^c f$$

$$= \int_a^b f + \int_c^a f$$

$$= \int_c^b f$$

$$\int_1^3 \sin(x)\,dx = -\cos(x)\Big|_1^3 = -\cos(3) - (-\cos(1))$$
$$= -\cos(3) + \cos(1))$$

Thm (Fundamental Theorem of Calculus I)

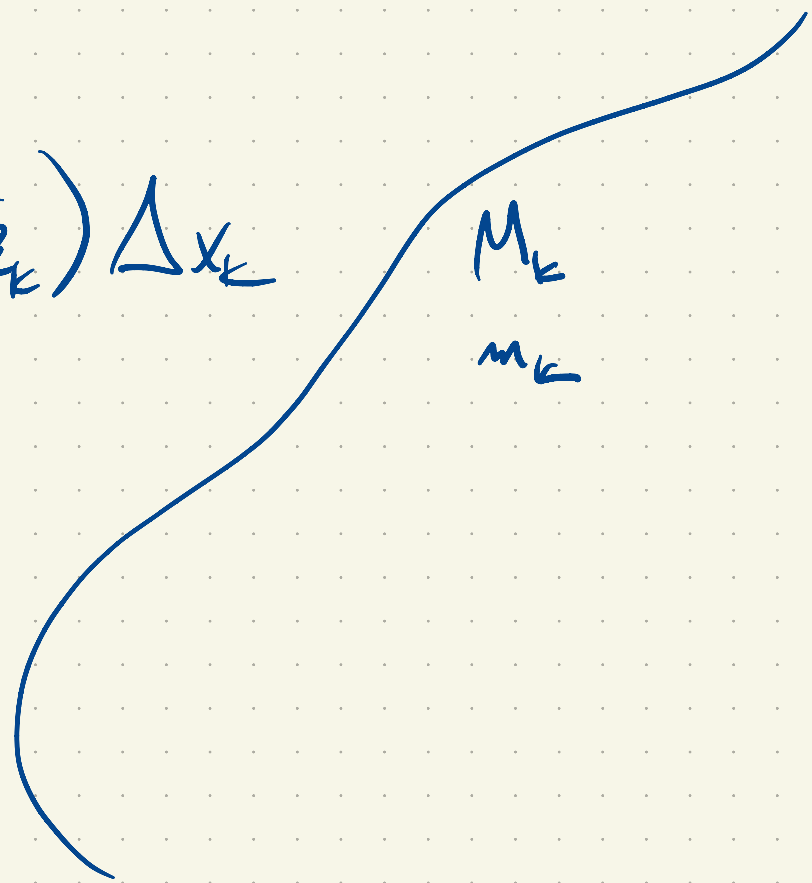
Suppose $f: [a,b] \to \mathbb{R}$ is Riemann integrable and $F: [a,b] \to \mathbb{R}$ is differentiable and $F'(x) = f(x)$ on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b .$$

Pf: Consider a partition $\mathcal{P}$ of $[a,b]$.

On subinterval k we can apply the Mean Value Theorem to conclude $\dfrac{F(x_k)-F(x_{k-1})}{x_k - x_{k-1}} = F'(\xi) = f(\xi_k)$ for some ξ_k in I_k. That is

$$F(x_k)-F(x_{k-1}) = f(\xi_k)\Delta x_k$$



for some $\xi_k \in I_k$. As a consequence,

$$L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} f(\xi_k) \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k$$

$$= U(f,P)$$

Note, however, that

$$\sum_{k=1}^{n} f(\xi_k) \Delta x_k = \sum_{k=1}^{n} \left(F(x_k) - F(x_{k-1}) \right)$$

$$= F(x_n) - F(x_0)$$

$$= F(b) - F(a).$$

So for any partition $\mathcal{P}$,

$$L(f,\mathcal{P}) \le F(b) - F(a) \le U(f,\mathcal{P}).$$

Taking a supremum and infimum over $\mathcal{P}$ we find

$$\int_a^b f \le F(b) - F(a) \le \int_a^b f.$$

$\square$

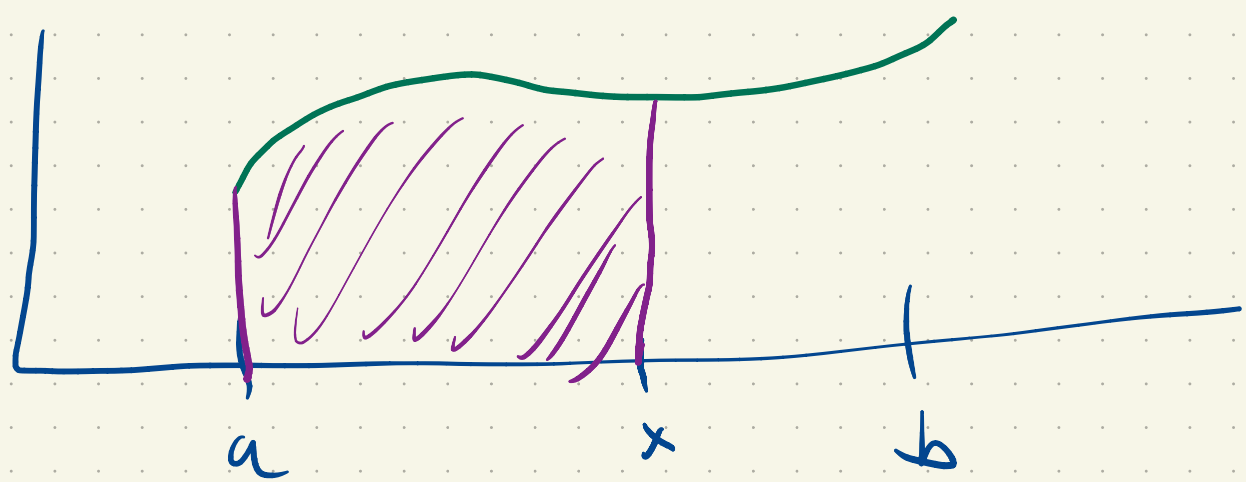
Part II (a partial converse).

g, Riemann integrable on $[a,b]$

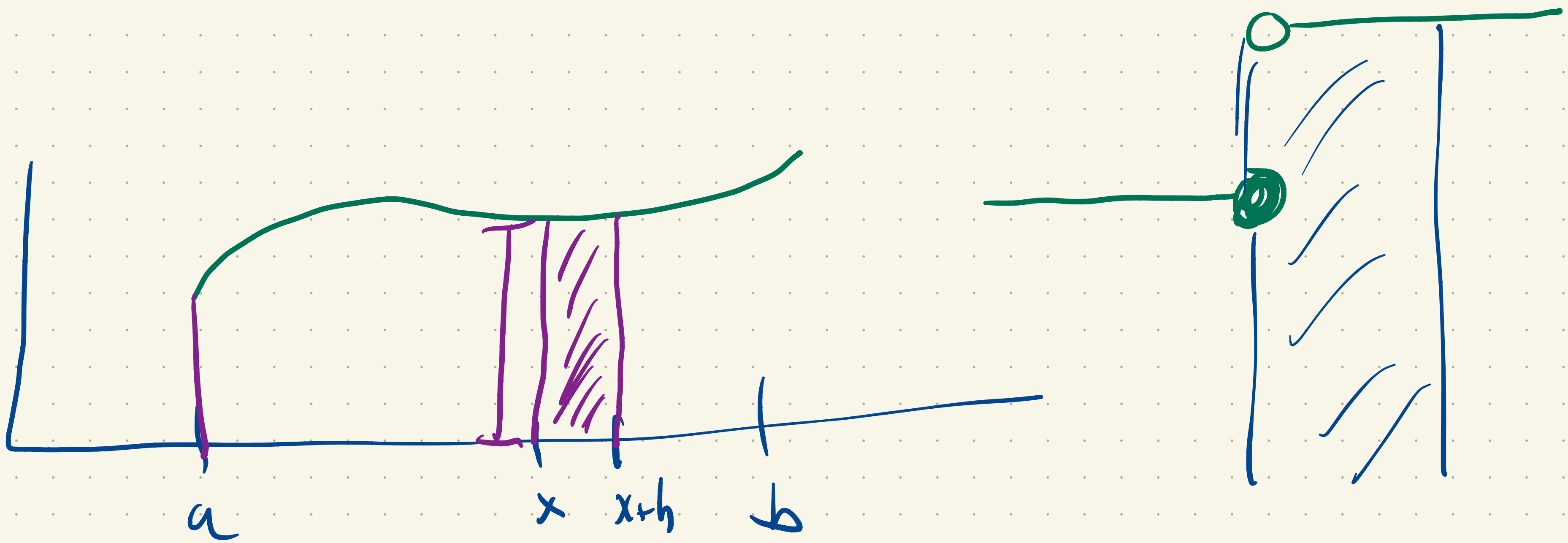
$$G(x) = \int_a^x g(s)\,ds$$

$$G(a) = \int_a^a g(s)\,ds = 0$$

$$G(b) = \int_a^b g(s)\,ds$$



$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$



$$\int_x^{x+h} g(s)\, ds \approx h\, g(x)$$

$$\frac{1}{h}\int_x^{x+h} g(s)\,ds \approx g(x)$$

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(s)\,ds = g(x)$$

$$\lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = g(x)$$

Morally: if you want an antiderivative for $g(x)$

just form $\int_a^x g(s)\,ds = G(x)$.

$$\frac{d}{dx} u(x) = e^{\sqrt{x}}$$

$$u(x) = \int_0^x e^{\sqrt{s}}\,ds + C$$

Thm (FTC II)

Suppose $g(x)$ is integrable on $[a,b]$ and define $G(x) = \int_a^x g(s)\,ds$ on $[a,b]$.

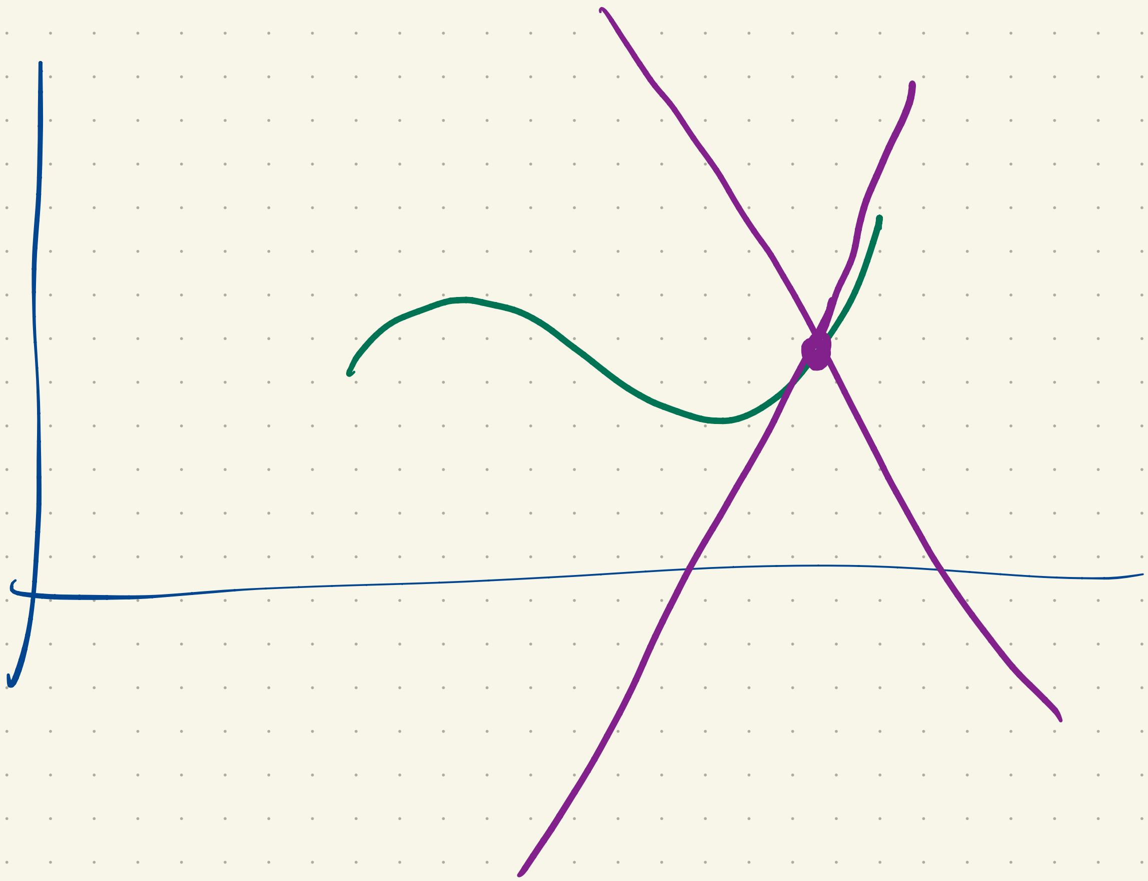
Then G is continuous and moreover, at any $c \in [a,b]$ where g is continuous

$$G'(c) = g(c).$$

In particular, if g is continuous on $[a,b]$ then $G' = g$.

Pf: We leave as an exercise the fact that G is Lipschitz continuous.

$$\Big[\, |G(x) - G(y)| \leq K|x-y| \,\Big] \quad \Big(\big||x|-|y|\big| \leq |x-y|\Big)$$



Suppose f is continuous at $c \in [a,b]$.

Let $\varepsilon > 0$. Pick $\delta > 0$ so that if $|x-c| < \delta$

then $|g(x) - g(c)| < \varepsilon$. This is possible since

g is continuous at c. Now observe that if

$0 < |x - c| < \delta$ then

$$\int_c^x g(c)\,ds = g(c)(x-c)$$

$$\frac{G(x) - G(c)}{x - c} - g(c) = \frac{1}{x-c}\int_c^x g(s)\,ds - g(c)$$

$$= \frac{1}{x-c}\int_c^x \big(g(s) - g(c)\big)\,ds.$$

But then, assume for the moment that $0 < |x-c| < \delta$ and $x > c$ then

$$\left| \frac{G(x) - G(c)}{x - c} - g(c) \right| = \left| \frac{1}{x-c}\int_c^x \big(g(s) - g(c)\big)\,ds \right|$$

$$\leq \frac{1}{x-c}\int_c^x |g(s)-g(c)|\,ds$$

$$\leq \frac{1}{x-c}\int_c^x \varepsilon\,ds$$

$$= \frac{x-c}{x-c}\cdot\varepsilon = \varepsilon.$$

A similar argument with sign changes happens when $x<c$ and we still find the same inequality holds. Consequently $\lim_{x\to c}\frac{G(x)-G(c)}{x-c} = g(c)$. $\square$