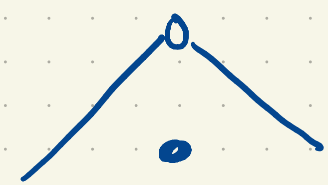
$$\int_a^b f(x)\,dx$$

$$f : [a,b] \to \mathbb{R}$$

bounded.

$$\int_0^1 \frac{1}{x^2}\,dx$$

$$U(f) = L(f) = \int_a^b f$$

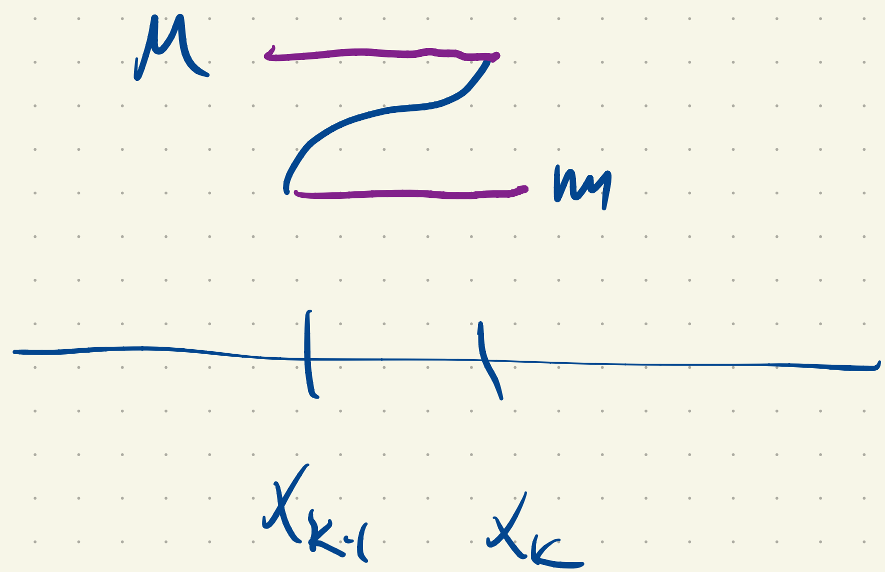


$$U(f,P) = \sum_{k=1}^{n} M_k \,\Delta x_k$$

$$\sup_{x \in I_k} f(x)$$

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$\Delta x_k = x_k - x_{k-1} \qquad L(f,P) = \sum m_k \,\Delta x_k$$



$$\mathcal{P}_1 \supseteq \mathcal{P}_2$$

$$U(f, \mathcal{P}_1) \leq U(f, \mathcal{P}_2)$$

$$\inf_{\mathcal{P}} U(f, \mathcal{P}) = U(f)$$

$$\sup_{\mathcal{P}} L(f, \mathcal{P}) = L(f)$$

$$\int_0^1 x \, dx$$

We'd like to show that all continuous functions on $[a,b]$ are Riemann integrable.

$\int_a^b f$ exists

---

bounded

Prop: A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon>0$ there exists a partition $\mathcal{P}$ such that $U(f,\mathcal{P})<L(f,\mathcal{P})+\varepsilon$.

$$\Downarrow$$

$$U(f,\mathcal{P})-L(f,\mathcal{P})<\varepsilon$$

$$\Downarrow$$

$$|U(f,\mathcal{P})-L(f,\mathcal{P})|<\varepsilon$$

Pf: Suppose f is R.I.

Let $\varepsilon > 0$. We can pick a partition $\mathcal{P}_1$ such that
$$U(f, \mathcal{P}_1) < U(f) + \varepsilon/2$$
and a partition $\mathcal{P}_2$ such that
$$L(f, \mathcal{P}_2) > L(f) - \varepsilon/2.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ be the common refinement.

Observe
$$L(f) - \frac{\varepsilon}{2} < L(f, \mathcal{P}_2) \leq L(f, \mathcal{P}) \leq L(f)$$
$$U(f) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{P}_1) < U(f) + \frac{\varepsilon}{2}.$$

Hence
$$U(f,P) < U(f) + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}$$
$$L(f,P) > L(f) - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2}.$$

Therefore
$$U(f,P) - L(f,P) < \varepsilon.$$

Conversely, suppose the main hypothesis of the prop holds.

Let $\varepsilon > 0$. Pick a partition P such that
$$U(f,P) < L(f,P) + \varepsilon.$$

Hence
$$U(f) \leq U(f,P) < L(f,P) + \varepsilon \leq L(f) + \varepsilon,$$

Consequently,

$$0 \leq U(f) - L(f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $U(f) = L(f)$.

Remark: if f is R.I., and If for some $\varepsilon > 0$ we find

$$U(f, P) < L(f, P) + \varepsilon$$

Then $$\int_a^b f \leq U(f, P) < \int_a^b f + \varepsilon$$

and $$\int_a^b f - \varepsilon < L(f, P) \leq \int_a^b f$$

Cor: If $f:[a,b]\to\mathbb{R}$ is continuous, then f is Riemann integrable

Pf: Let $\varepsilon>0$. Since $[a,b]$ is compact, f is uniformly continuous. We can therefore pick a $\delta>0$ so that if $x,y\in[a,b]$ and $|x-y|<\delta$ then $|f(x)-f(y)|<\varepsilon/_{b-a}$.

Let $\mathcal{P}$ be a partition such that each $\Delta x_k<\delta$.

$$\downarrow$$

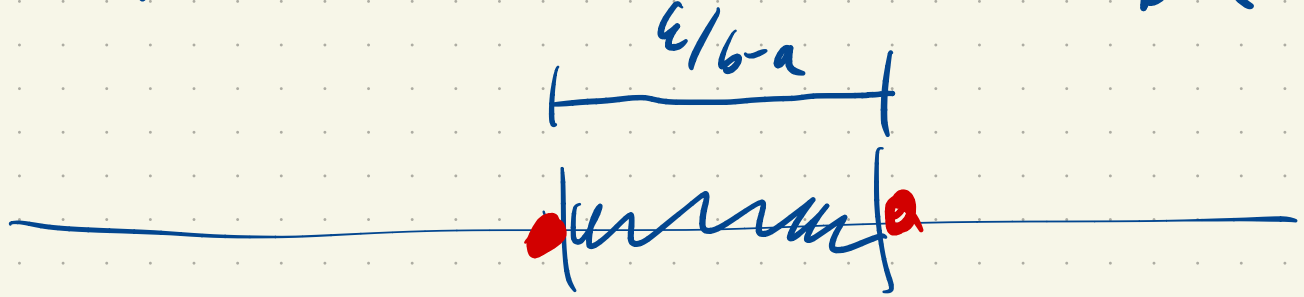
$$\frac{b-a}{n}<\delta$$

Observe that on each subinterval I_k,

$$M_k = \sup_{x \in I_k} f(x) \le \inf_{x \in I_k} f(x) + \frac{\varepsilon}{b-a} = m_k + \frac{\varepsilon}{b-a}.$$

$f(x^*)$ $\qquad$ $f(x_*)$

$z, w \in f(I_k)$ $\qquad$ $|z-w| < \frac{\varepsilon}{b-a}$



$f(I_k)$

$$\boxed{M_k \leq m_k + \frac{\varepsilon}{b-a}}$$

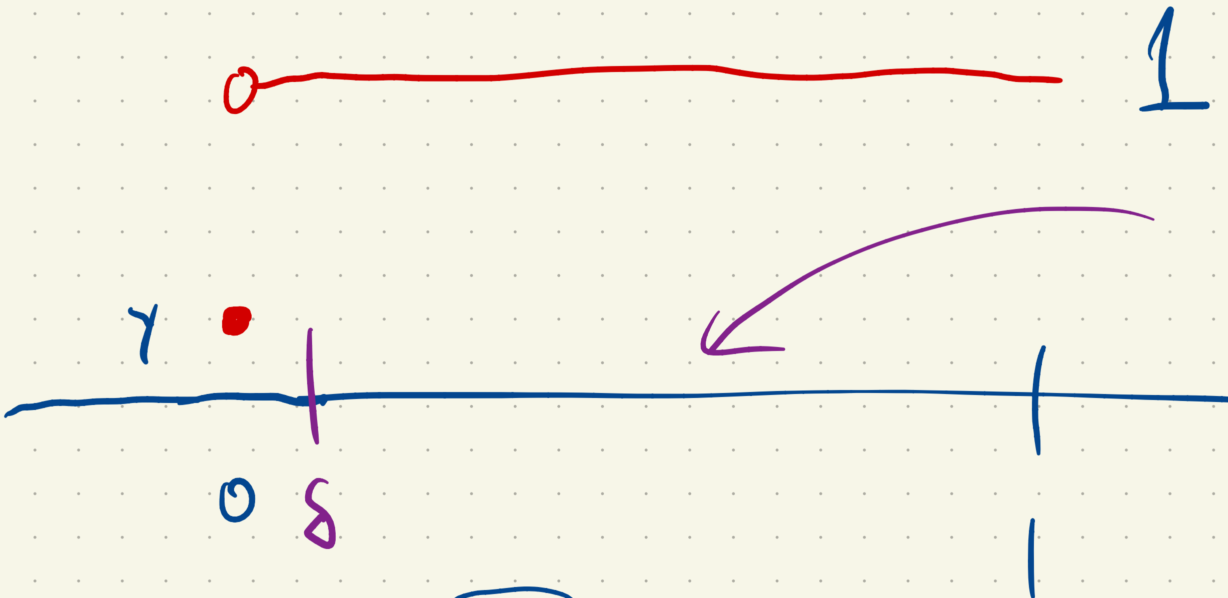
Observe

$$U(f,P) = \sum_{k=1}^{n} M_k \Delta x_k \leq \sum_{k=1}^{n}\left(m_k + \frac{\varepsilon}{b-a}\right)\Delta x_k$$

$$= L(f,P) + \frac{\varepsilon}{b-a}\sum_{k=1}^{n}\Delta x_k$$

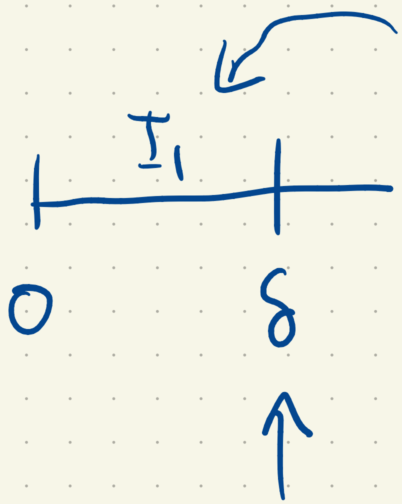
$$= L(f,P) + \varepsilon$$

as desired. $\square$



Pick a great partition over here.

$|f(x)| \le M$



$M_1 \Delta x_1 \le M \Delta x_1$

$m_1 \Delta x_1 \ge -M \Delta x_1$

$U(f, P) < L(f, P) + \frac{\varepsilon}{2}$

Great partition P of $[\delta, 1]$

$\mathcal{P}^* = \{0\} \cup \mathcal{P}$. (partition of $[0,1]$)

$$U(f, \mathcal{P}^*) = M_1 \Delta x_1 + U(f, \mathcal{P}) \leq M\delta + U(f, \mathcal{P})$$

$$L(f, \mathcal{P}^*) = m_1 \Delta x_1 + L(f, \mathcal{P}) \geq -M\delta + L(f, \mathcal{P})$$

$$-L(f, \mathcal{P}^*) \leq M\delta - L(f, \mathcal{P})$$

$$U(f, \mathcal{P}^*) - L(f, \mathcal{P}^*) \leq 2M\delta + U(f, \mathcal{P}) - L(f, \mathcal{P})$$

$$< 2M\delta + \varepsilon/2.$$

We could have picked $\delta < \frac{\varepsilon}{4M}$.

$$2M\delta < \varepsilon/2$$

Prop: Suppose $f : [a,b] \to \mathbb{R}$ is bounded and that f is Riemann integrable on $[c,b]$ with $a<c<b$. Then f is Riemann integrable on $[a,b]$ as well.