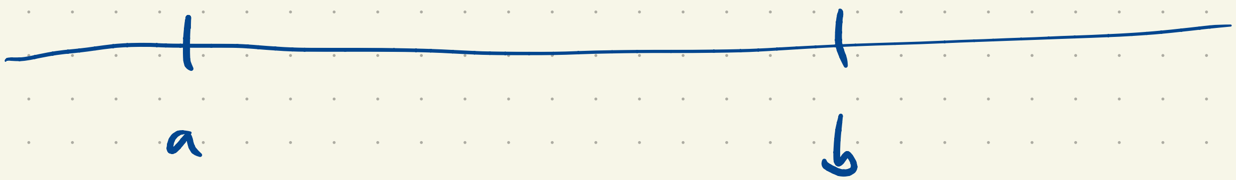
$f : [a,b] \to \mathbb{R}$

bounded

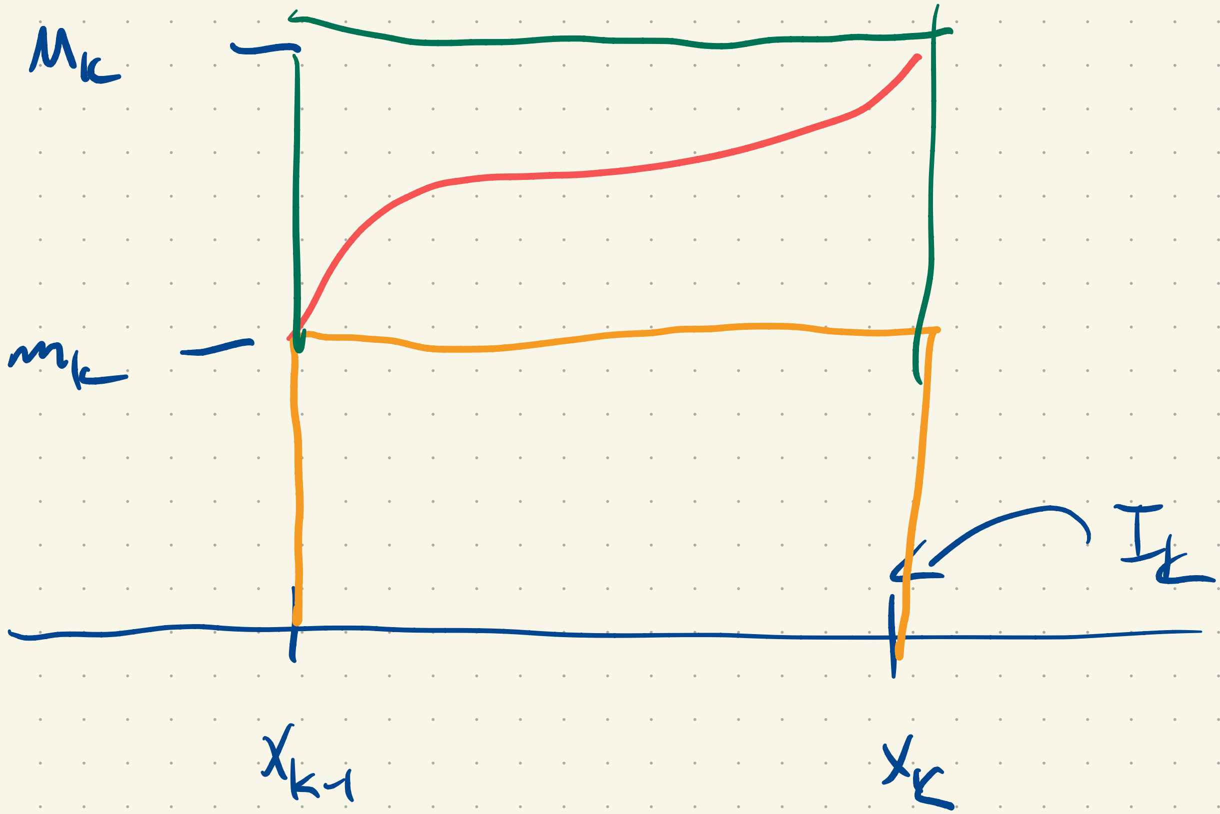
$$\int_a^b f(x)\,dx$$



$P \to$ finite subset of $[a,b]$ that contains a, b.

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

$$I_k = [x_{k-1}, x_k], \quad \Delta x_k = x_k - x_{k-1}$$



$$M_k = \sup \{ f(x) : x \in I_k \}$$

$$= \sup_{x \in I_k} f(x)$$

$$m_k = \inf \{ f(x) : x \in I_k \}$$

$$U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k$$ upper sum

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k$$ lower sums.

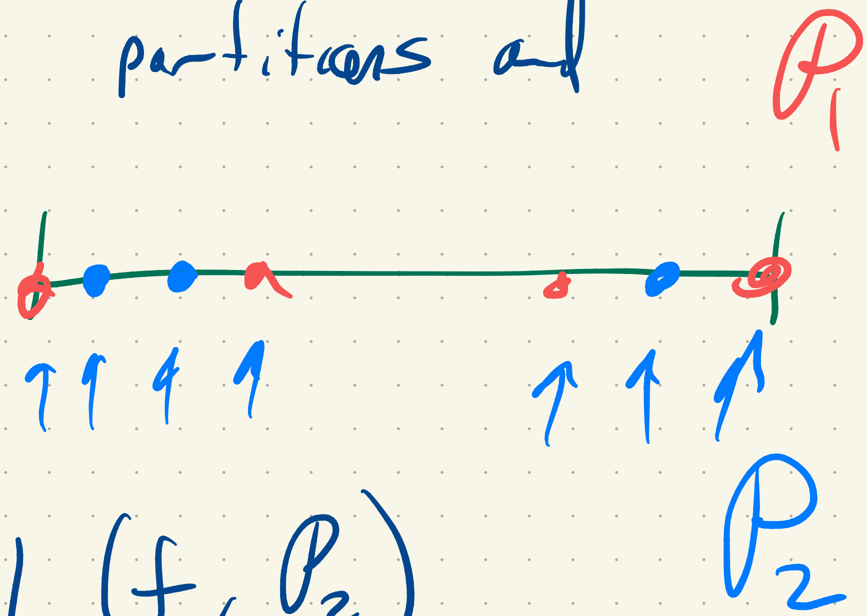
Morally:

$$L(f, P) \leq \int_a^b \underbrace{f(x)\,dx} \leq U(f, P)$$

$$\int_a^b f$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions and

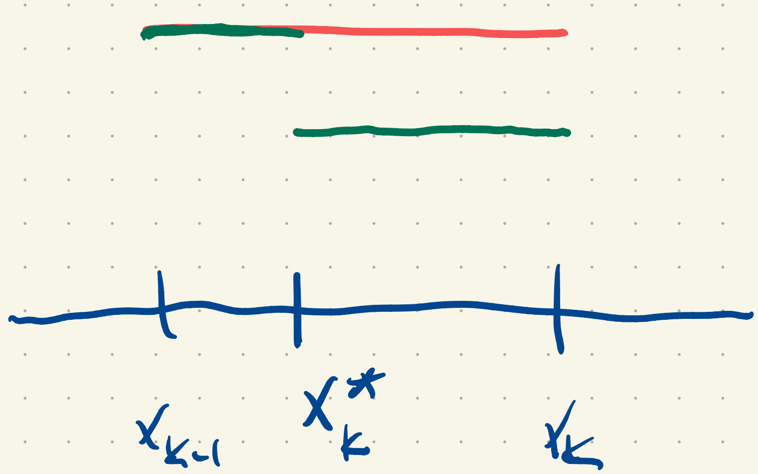
$$\mathcal{P}_2 \supseteq \mathcal{P}_1$$



then

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_2)$$

$$U(f, \mathcal{P}_1) \geq U(f, \mathcal{P}_2)$$



Given partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, their <u>common</u> <u>refinement</u> is $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$

$$\mathcal{P} \supseteq \mathcal{P}_1 \qquad \mathcal{P} \supseteq \mathcal{P}_2$$

$$L(f,\mathcal{P}_2) \leq L(f,\mathcal{P}) \leq U(f,\mathcal{P}) \leq U(f,\mathcal{P}_1)$$

$$L(f,\mathcal{P}_2) \leq U(f,\mathcal{P}_1)$$

for <u>all</u> partitions $\mathcal{P}_1, \mathcal{P}_2$.

Each partition yields an estimate from above for $\int_a^b f$; $U(f,P)$.

Our best estimate from above:

$$\underbrace{\inf\{U(f,P) : P \text{ is a partition of } [a,b]\}}_{U(f)}$$

$\longrightarrow U(f) \longrightarrow$ upper integral

lower integral

$$L(f) = \sup\{L(f,P) : P \text{ is a } \ldots [a,b]\}$$

Is $\{U(f,P) : P$ is a part of $[a,b]\}$

bounded below? Nonempty?

$$P = \{a, b\} \quad U(f,P).$$

For a lower bound we could use

$$L(f, \{a,b\}) \leq U(f,P)$$

$$\forall P.$$

$$L(f) \qquad\qquad U(f)$$

$$L(f, P_1) \leq U(f, P_2)$$

$$L(f, P_1) \leq U(f)$$

$$L(f) \leq U(f)$$

Bad news: strict inequality is possible.

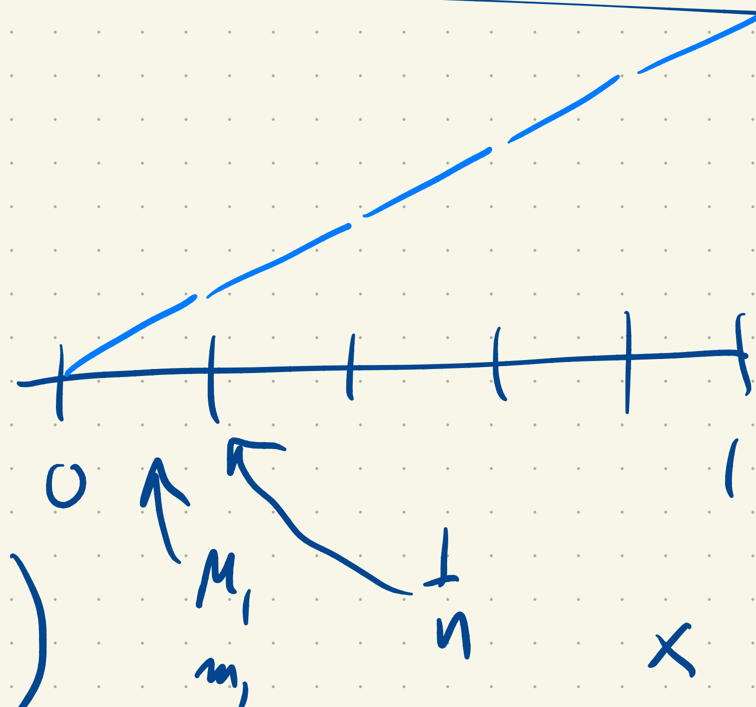
Def: We say a function $f : [a,b] \to \mathbb{R}$
is Riemann integrable if

$L(f) = U(f)$, in which case

$\int_a^b f$ is the common value.

$f(x) = x$ on $[0,1]$

$\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$



$U(f, \mathcal{P}_n)$ $\quad\quad$ $L(f, \mathcal{P}_n)$

$I_k = [x_{k-1}, x_k]$ $\quad\quad$ $\Delta x_k = ?$

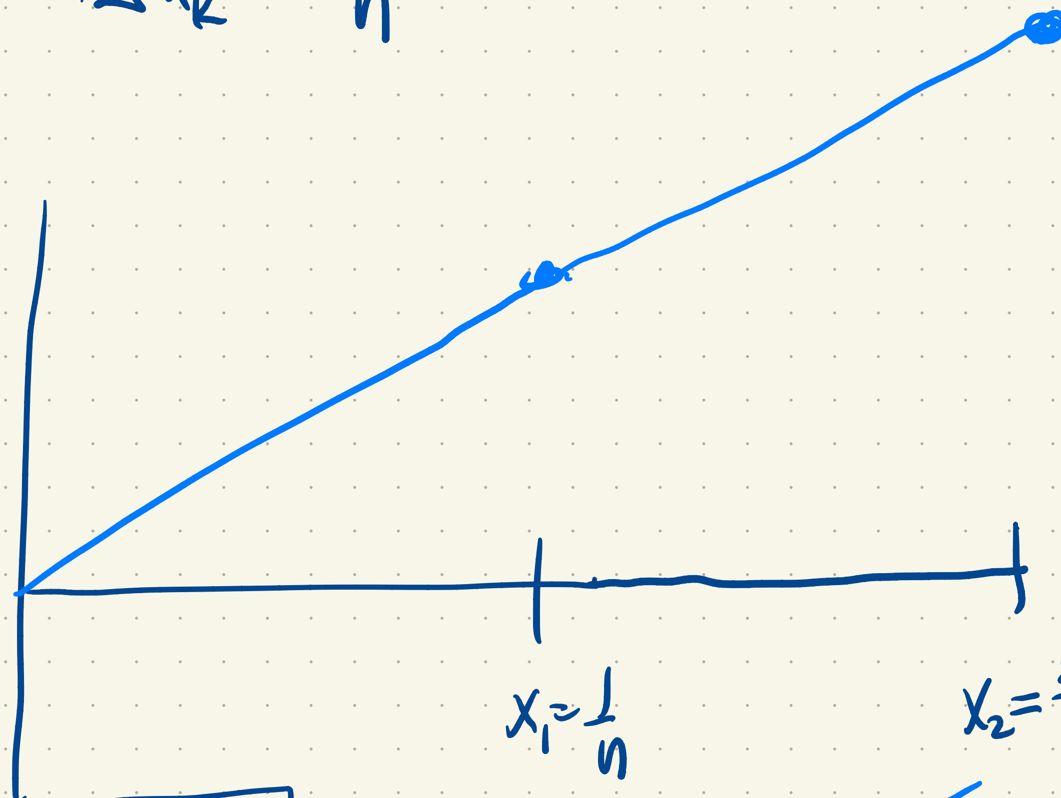
$M_1 =$

$m_1 =$

$$\left[\frac{k-1}{n}, \frac{k}{n}\right] \qquad \Delta x_k = \frac{1}{n}$$

$$M_1 = \frac{1}{n}$$

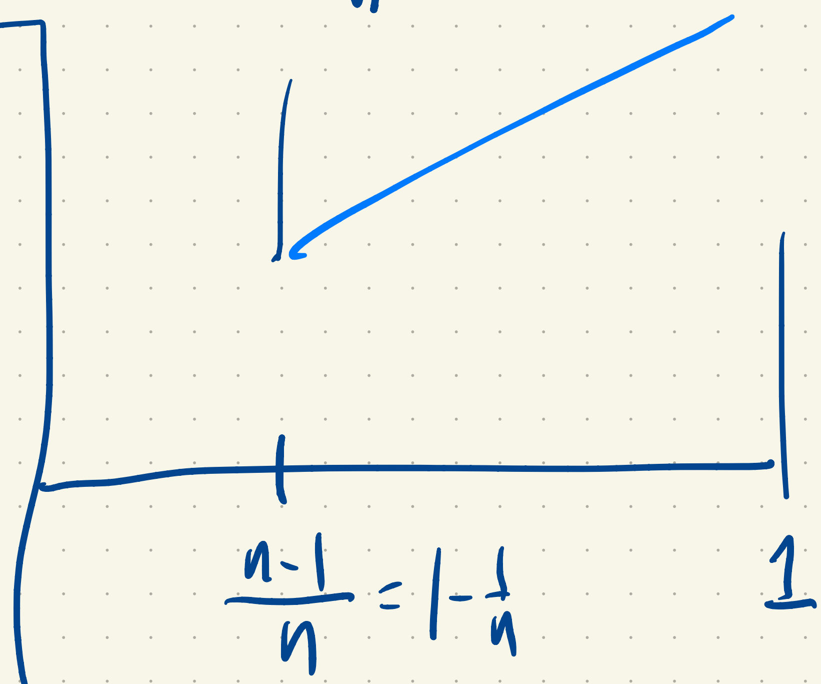
$$m_1 = 0$$



$$M_2 = \frac{2}{n}$$

$$m_2 = \frac{1}{n}$$

$$\boxed{\begin{aligned} M_k &= \frac{k}{n} \\ m_k &= \frac{k-1}{n} \end{aligned}}$$



$$M_n = 1$$

$$m_n = \frac{n-1}{n}$$

$$U(f, P_n) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} \frac{k}{n} \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} k$$

$$= \frac{1}{n^2} \frac{n(n+1)}{2}$$

$$= \frac{1}{2} + \frac{1}{2n}$$

$$1 + 2 + 3 + \cdots + n$$

$$n + n-1 + \cdots + 1$$

$$\underline{(n+1)} + \underline{(n+1)} + \cdots + \underline{(n+1)} = n(n+1)$$

$$M_k = m_k + \frac{1}{n}$$

$$\begin{aligned}
L(f, P_n) = \sum_{k=1}^{n} m_k \Delta x_k &= \sum_{k=1}^{n} \left(M_k - \frac{1}{n}\right) \Delta x_k \\
&= \sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} \frac{1}{n} \Delta x_k \\
&= U(f, P_n) - \sum_{k=1}^{n} \frac{1}{n^2} \\
&= U(f, P_n) - \frac{1}{n} \\
&= \frac{1}{2} - \frac{1}{2n}
\end{aligned}$$

$$\overbrace{\frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2}}^{n}$$

$\mathcal{P}_n$

$$U(f, \mathcal{P}_n) = \frac{1}{2} + \frac{1}{2n}$$

$$L(f, \mathcal{P}_n) = \frac{1}{2} - \frac{1}{2n}$$

$$U(f) = \inf\{U(f, \mathcal{P}) ; \mathcal{P} \text{ is a part}\}$$

$$\inf\{U(f, \mathcal{P}_n) : n\}$$

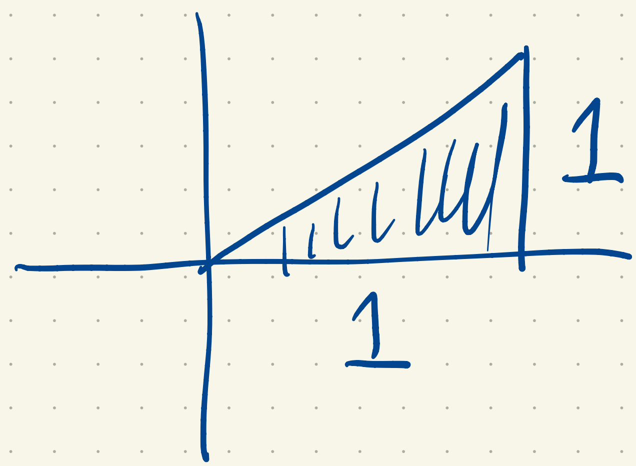
$$U(f) \leq \inf_n U(f, \mathcal{P}_n) = \inf_n \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}$$

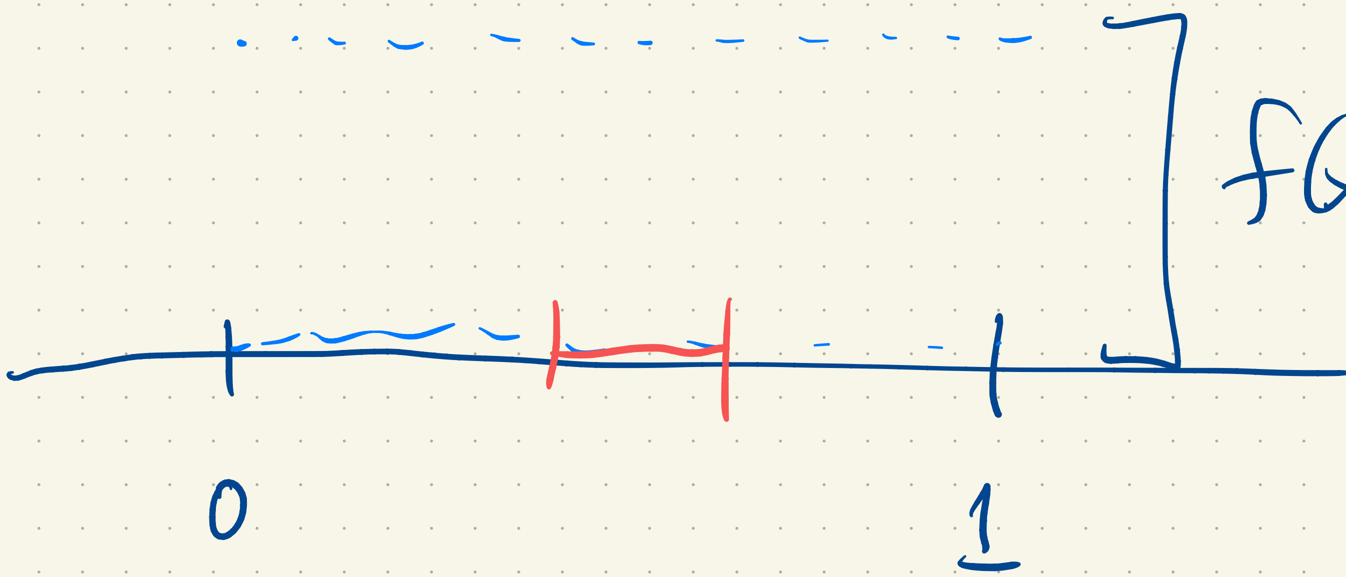
$$L(f) \geq \sup_n L(f, P_n) = \sup_n \frac{1}{2} - \frac{1}{2n} = \frac{1}{2}$$

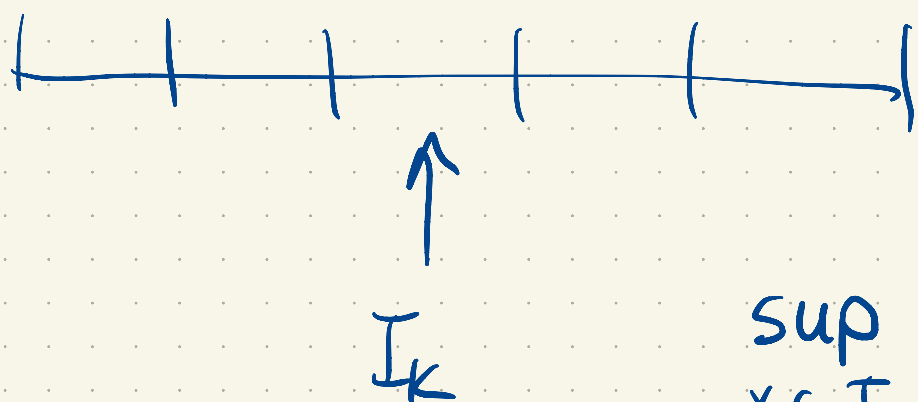
$$\frac{1}{2} \leq \underbrace{L(f) \leq U(f)}_{} \leq \frac{1}{2}$$

$$L(f) = U(f) = \frac{1}{2}$$

$$\int_0^1 x\,dx = \frac{1}{2}$$







$$\sup_{x \in I_k} f(x) = 1 = M_k$$

$$\inf_{x \in I_k} f(x) = 0 = m_k$$

$$U(f,P) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} 1 \, \Delta x_k$$

$$= \sum_{k=1}^{n} \Delta x_k = 1$$

$$L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} 0 \, \Delta x_k = 0$$

$$U(f) = \inf \{ U(f,P) : P \} = 1$$

$$L(f) = \sup \{ L(f,P) : P \} = 0$$

$$U(f) > L(f)$$

f is <u>not</u> Riemann integrable and we decline to define $\int_0^1 f$.