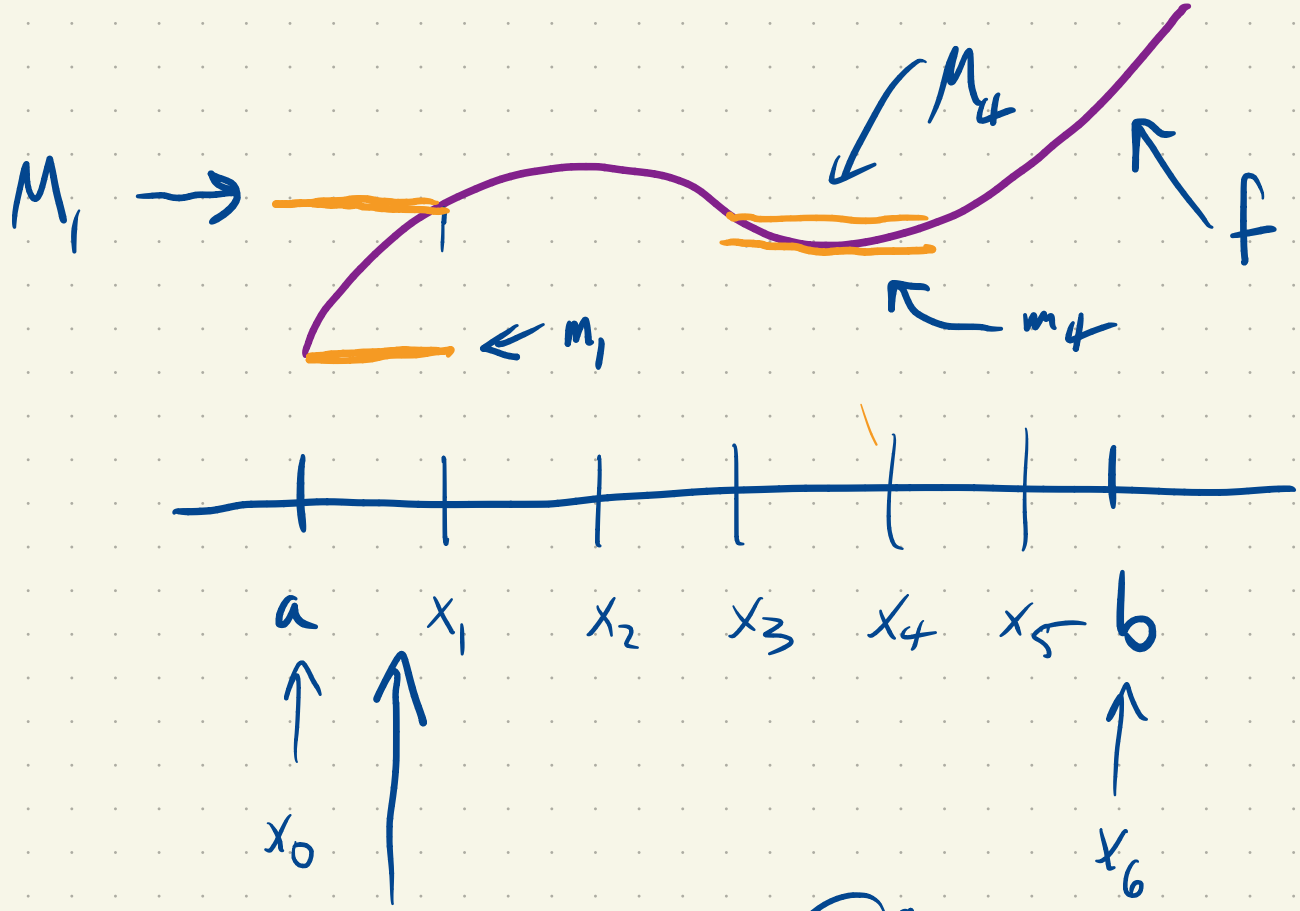


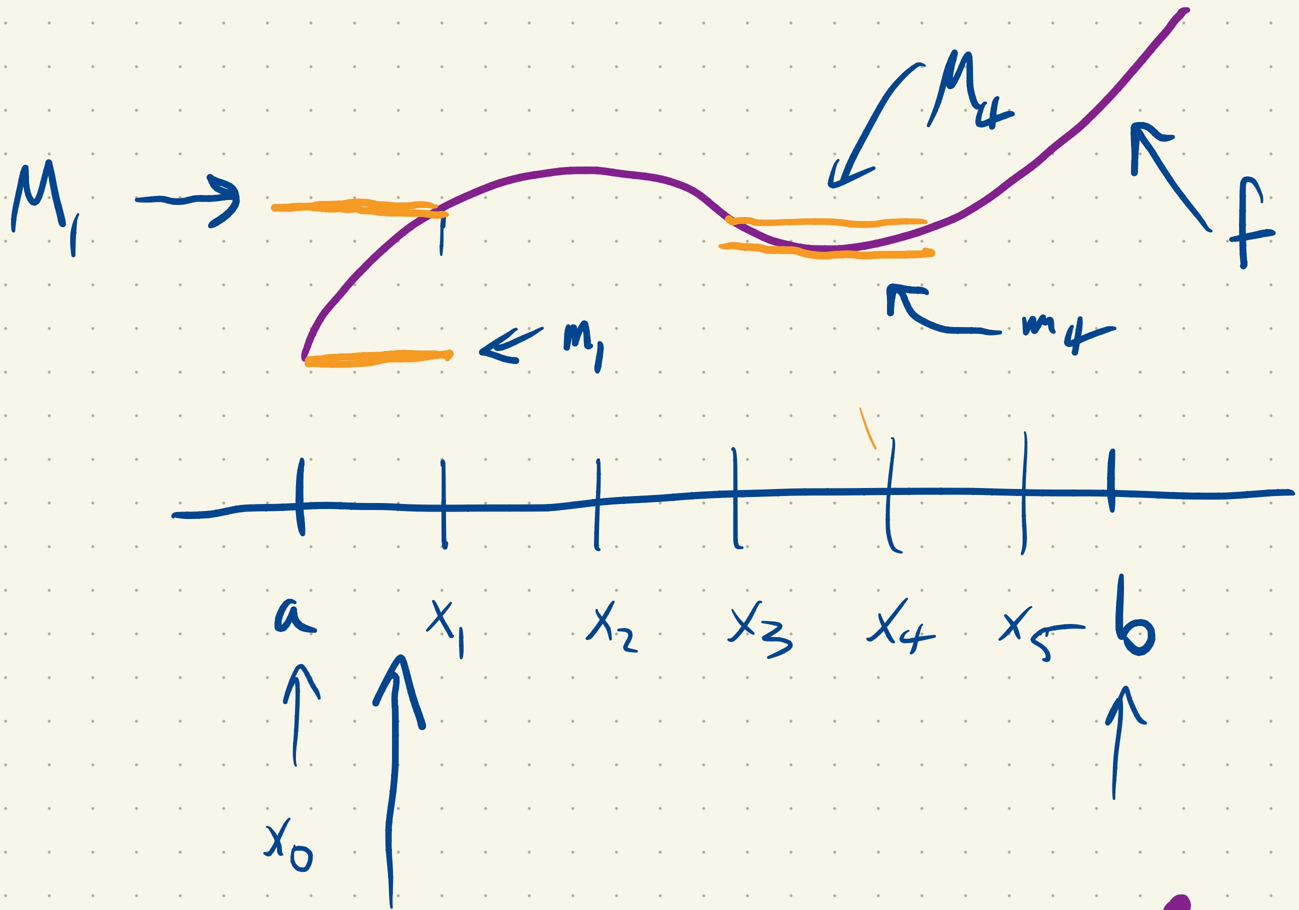
$$\sum_{k=1}^{6} M_k \Delta x_k$$

$$\sum_{k=1}^{6} m_k \Delta x_k$$

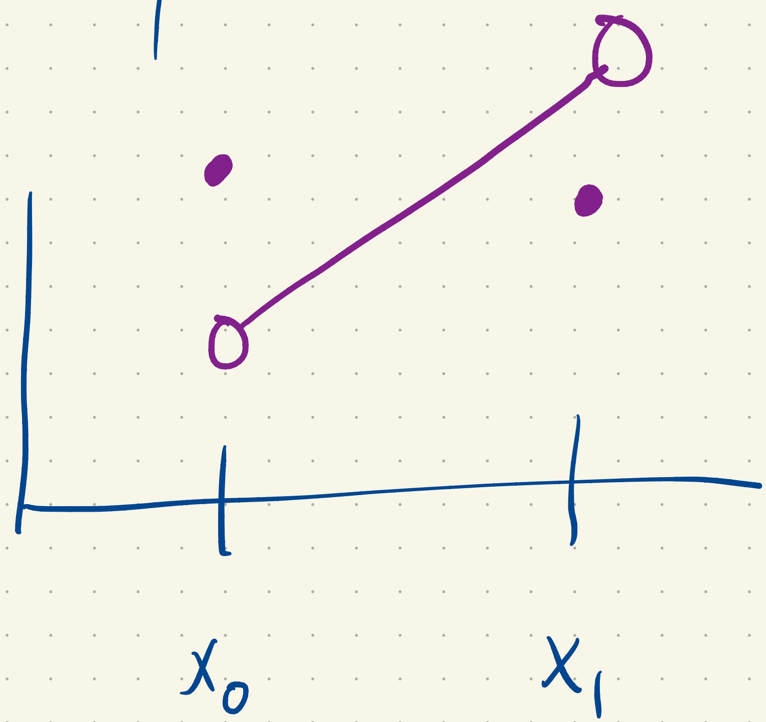
a

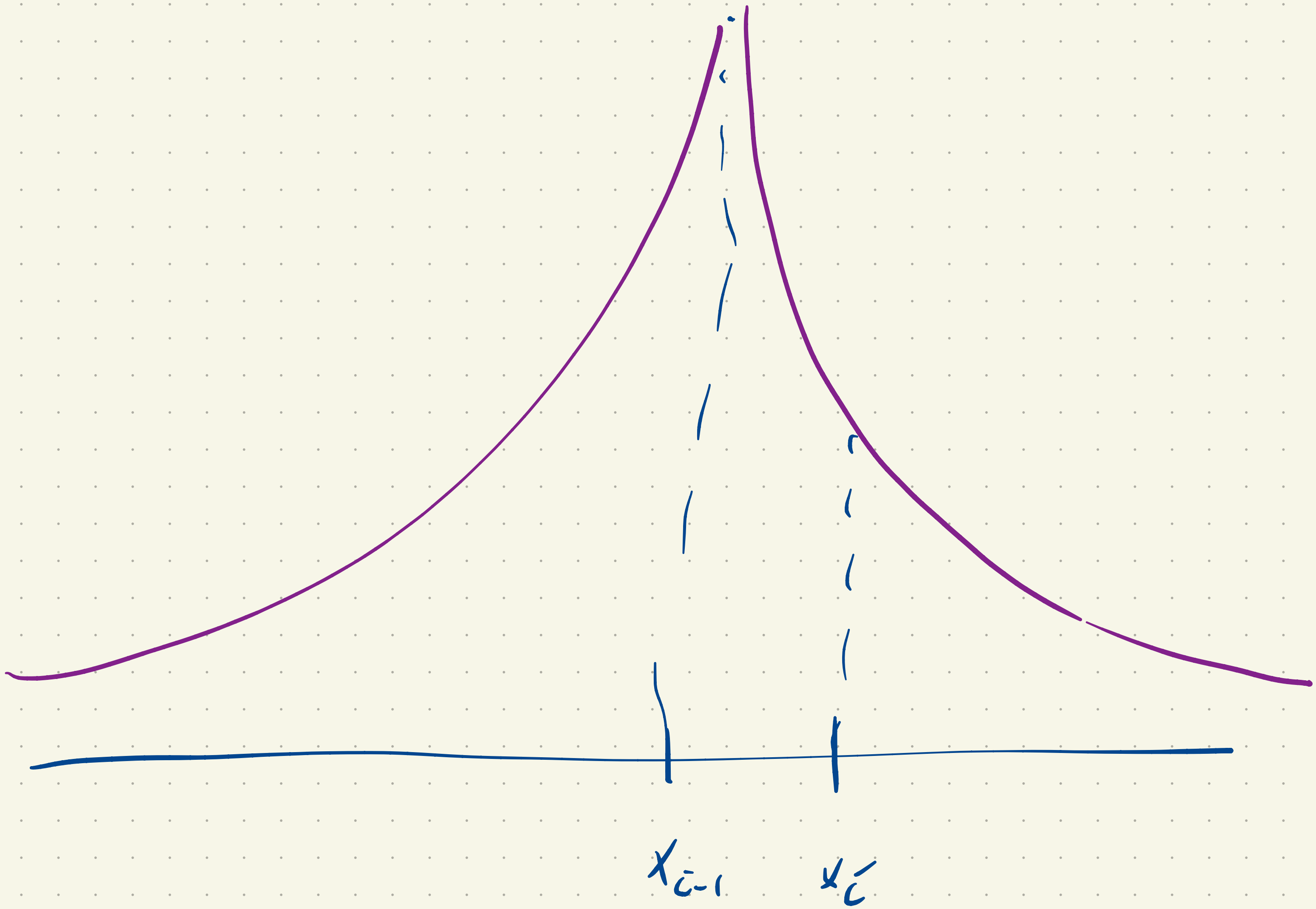
$$m_1 \Delta x_1 \leq A_1 \leq M_1 \cdot \underbrace{(x_1 - x_0)}_{\Delta x_1}$$

$$\Delta x_i = x_i - x_{i-1}$$



$$M_1 = \max_{x \in [x_0, x_1]} f(x)$$





We'll assume f is bounded.

I.e. there exists $M > 0$ such that

$$-M \le f(x) \le M$$

$$\left\{ f(x) : \overbrace{x_{i-1} \le x \le x_i}^{\Delta x_i \neq 0} \right\}$$

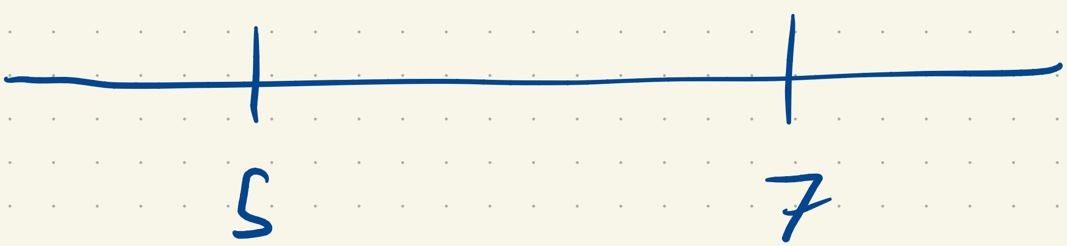
This set is bounded above (by M) and below by $-M$.

The set is nonempty if $\Delta x_i \neq 0$

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = \sup \{ f(x) : x_{k-1} \le x \le x_k \}$$

m_k is same but with inf.

Def: A <u>partition</u> of an interval $[a,b]$ is a finite subset that contains a and b.



$$\{7, 6.2, 5, 6.8\}$$ $$[5, 7]$$

$$\mathcal{P} = \{x_k\}_{k=0}^{n}$$

$x_0 = 5$, $x_1 = 6.2$

$x_3 = 6.8$, $x_4 = 7$

$$x_0 = a, \quad x_n = b$$

$$x_0 < x_1 < x_2 < \cdots < x_n$$

We have a partition $\mathcal{P}$ of $[a, b]$

$\hookrightarrow \{x_k\}_{k=0}^{n}$

$$I_k = [x_{k-1}, x_k] \quad 1 \leq k \leq n$$

Suppose $f : [a,b] \to \mathbb{R}$ is bounded.

$$M_k = \sup_{x \in I_k} f(x)$$

$$m_k = \inf_{x \in I_k} f(x)$$

Def: Given a partition P of $[a,b]$, the <u>upper sum</u> of a bounded function $f : [a,b] \to \mathbb{R}$

is (using the notation from above)

$$U(f,P) = \sum_{k=1}^{n} M_k \underbrace{(x_k - x_{k-1})}_{\Delta x_k} = \sum_{k=1}^{n} M_k \Delta x_k .$$

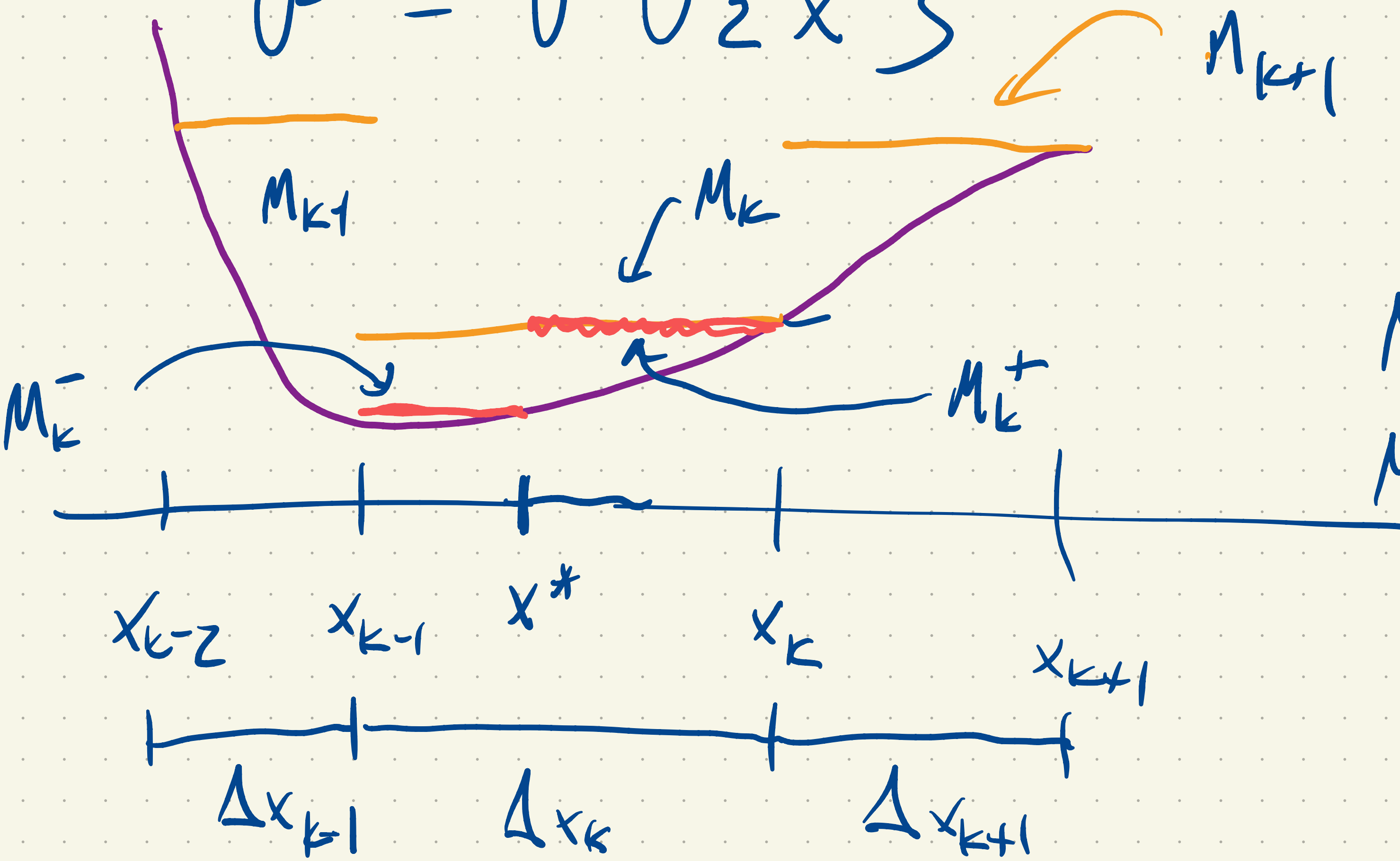
The <u>lower sum</u> is

$$L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k .$$

$$L(f,P) \leq U(f,P)$$

What happens to $U(f, P)$ if I add one new point to P.

$$P' = P \cup \{x^*\}$$



M_k^- vs M_k

M_k^+ vs M_k

$$\cdots + M_{k-1}\Delta x_{k-1} + \underbrace{M_k \Delta x_k} + M_{k+1}\Delta x_{k+1} + \cdots$$

$$\Big\downarrow$$

$$M_k^-(x^* - x_{k-1}) + M_k^+(x_k - x^*)$$

$$M_k = \sup \overbrace{\{ f(x) : x_{k-1} \leq x \leq x_k \}}^{F_k}$$

$$M_k^- = \sup \overbrace{\{ f(x) : x_{k-1} \leq x \leq x_* \}}^{F_k^-}$$

F_k^- vs F_k $\qquad$ $F_k^- \subseteq F_k$

$$M_k \geq M_k^-$$

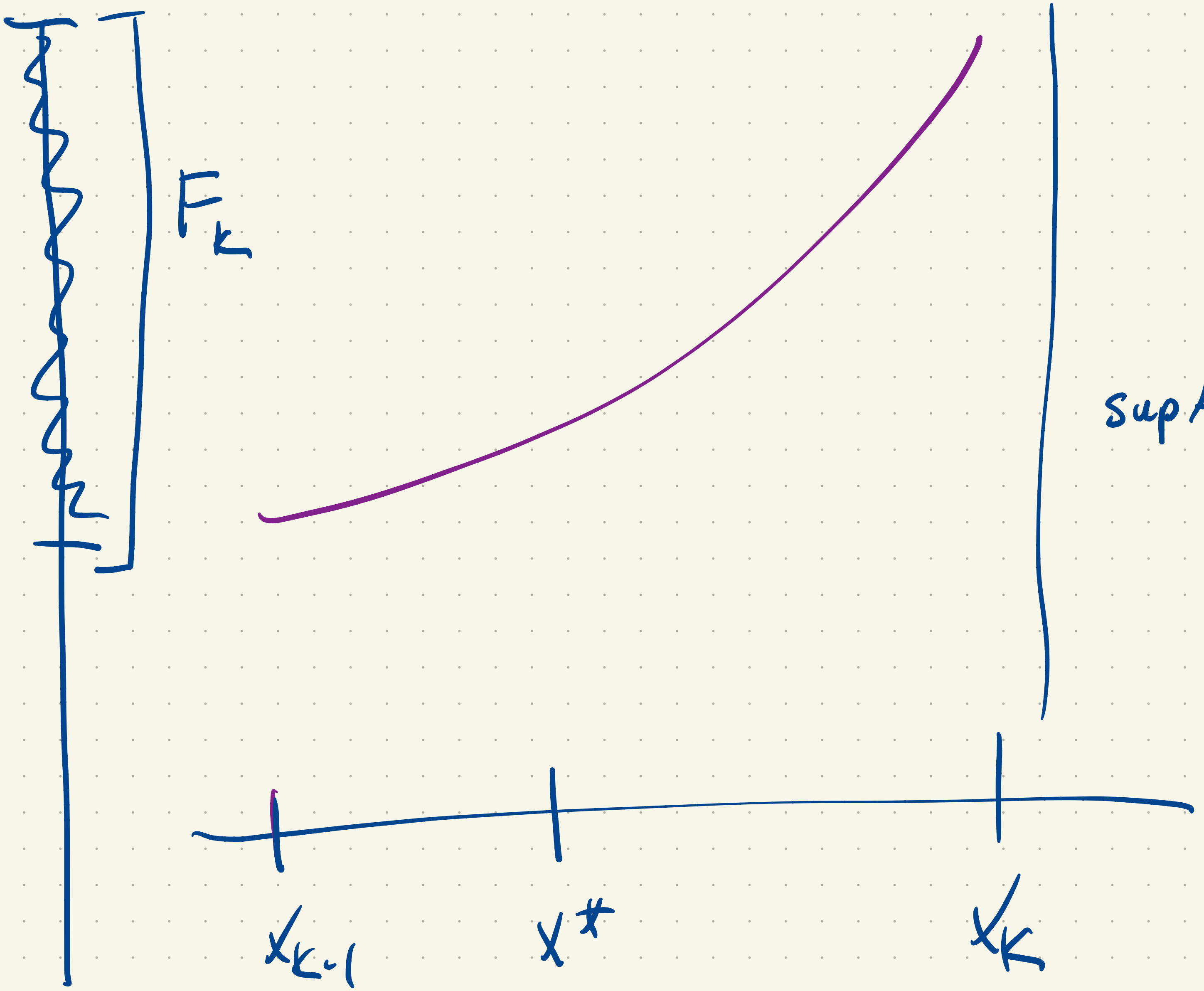
$$M_k \geq M_k^+$$

$$\underline{M_k^-(x^* - x_{k-1}) + M_k^+(x_k - x^*)} \leq M_k(x^* - x_{k-1})$$

$$+ M_k(x_k - x_*)$$

$$= M_k(x_* - x_{k-1} + x_k - x_*)$$

$$= \underline{M_k(x_k - x_{k-1})}$$



A, B, bounded above, $\neq\emptyset$

$A \subseteq B$

$\sup A \leq \sup B$

$A \subseteq B$, $\neq \emptyset$, bounded above

$\sup A \leq \sup B$

1) It is an upper bound for A

2) If y is any upper bound for A, $\sup A \leq y$.

1) It is an upper bound for B

2) If x is any upper bound for B, $\sup B \leq x$.

$\sup A \leq \sup B$

If $A \subseteq B$ and if x is an upper bound for B, x is also an u.b. for A.

$$U(f, P) \geqslant U(f, P \cup \{x_*\})$$

$$P_1 \subseteq P_2$$

$$U(f, P_2) \leq U(f, P_1)$$

$P_1 , \quad P_2$ $\qquad\qquad$ $P_1 \subseteq P_2$

$P_2 \subseteq P_1$

$L(f, P_1)$ $\qquad\qquad$ $U(f, P_2)$

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \quad \text{(common refinement)}$$

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{P}_2)$$

$$\underbrace{L(f, \mathcal{P}_1)}_{\text{supremum over partitions}} \qquad \int_a^b f(x)\,dx \qquad \underbrace{U(f, \mathcal{P}_2)}_{\text{inf over partitions}}$$

$$=$$