Mean Value Theorem:

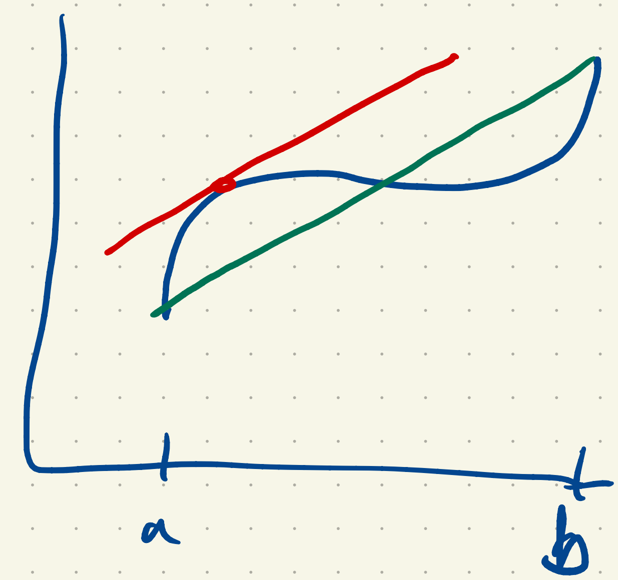
$f: [a,b] \to \mathbb{R}$, continuous.

f is diff on (a,b).

Then there exists $c \in (a,b)$

such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Same setup, but you know $f'(x) = 0$

for all $x \in (a,b)$. $\Rightarrow$ f is constant.

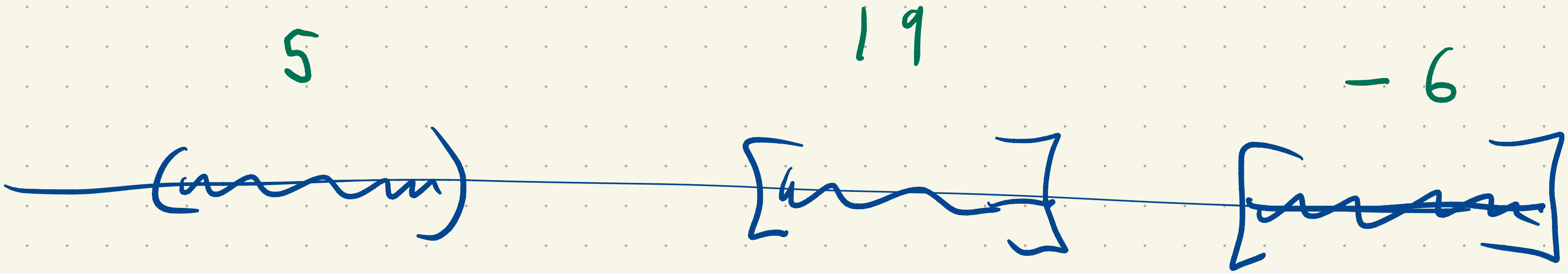
$[a, x]$ $\quad a < x \leq b$

$(a, x) \subseteq (a,b)$

There exists $c \in (a, x)$ such that

$$0 = f'(c) = \frac{f(x) - f(a)}{x - a}. \Rightarrow f(x) = f(a).$$

$f(x) = f(a) \quad \forall x \in (a, b]$ $\quad [a,b]$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$f'(x) > 0 \quad (a,b)$

If $\quad a \leq x_1 < x_2 \leq b$

$$\Rightarrow f(x_2) > f(x_1)$$

I.e. f is strictly increasing.

Prop: Suppose f is continuous on $[a,b]$, differentiable on (a,b) and $f'(x) > 0$ $\geq$ for all $x \in (a,b)$.

Then if $x_1, x_2 \in [a,b]$ and $x_1 < x_2$

then $f(x_1) < f(x_2)$.

$\leq$

Pf: Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$.

Observe f is continuous on $[x_1, x_2]$

and differentiable on $(x_1, x_2) \subseteq (a,b)$.

The mean value theorem then implies

there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Hence $f(x_2) = f(x_1) + f'(c)(x_2 - x_1)$.

Since $f'(c) > 0$ and since $x_2 > x$

we conclude $f'(c)(x_2 - x_1) > 0$ and

$$f(x_2) > f(x_1).$$

$\square$

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f, g on $[a,b]$, cts,

diff (a,b)

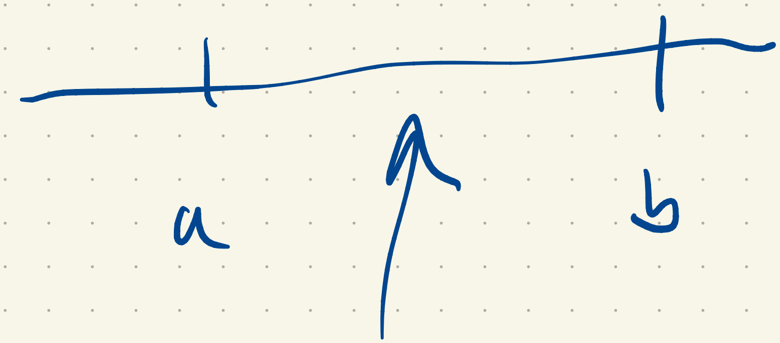
$f'(x) = g'(x) \quad \forall x \in (a,b)$

$\forall x\ (a,b)$

$F(x) = f(x) - g(x) \qquad F'(x) = f'(x) - g'(x) = 0$

F is constant

$F(x) = c \quad \forall x$

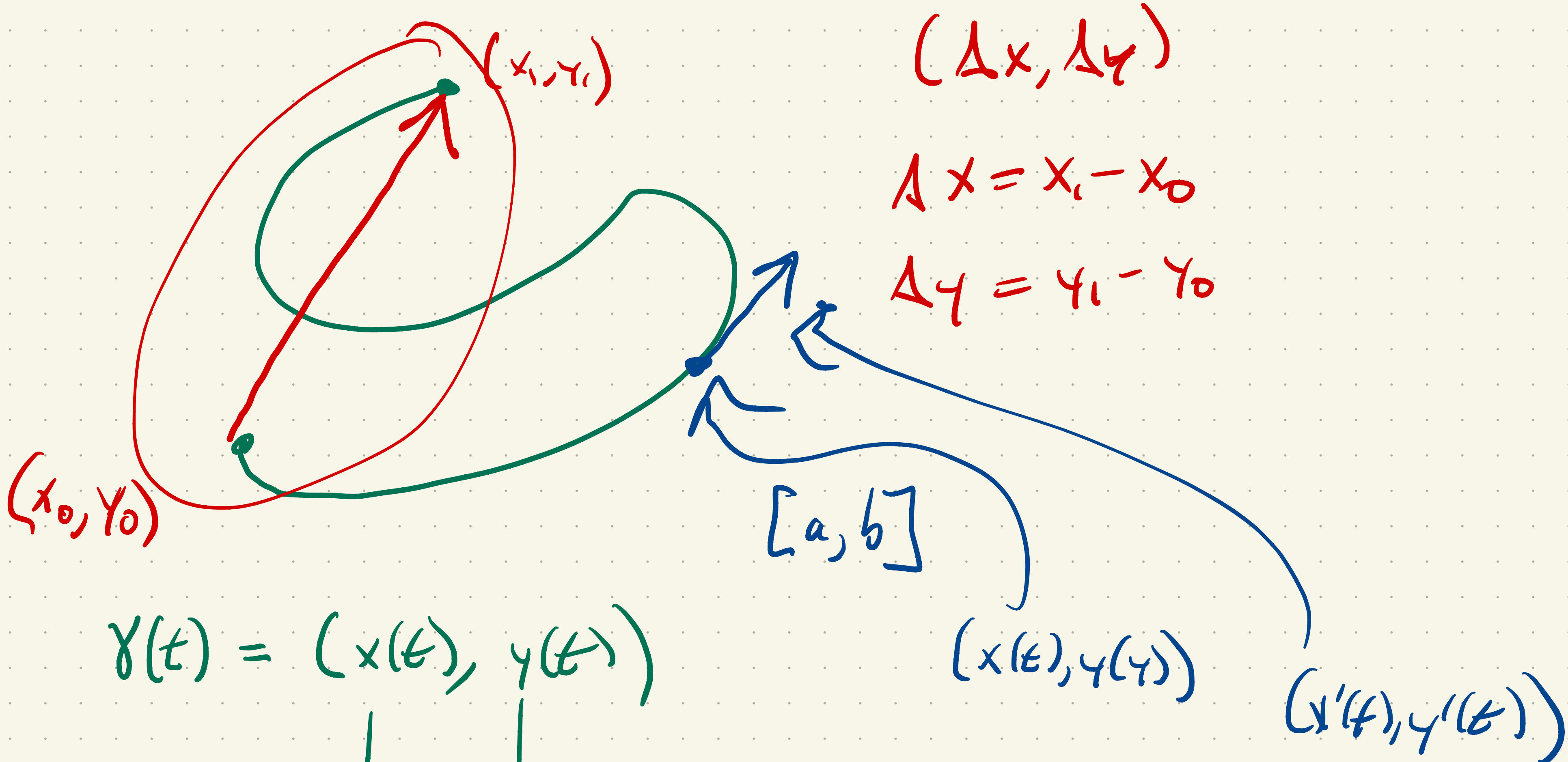


$f(x) - g(x) = c \quad \forall x$

$F'(x) = 0$

everywhere

$f(x) = g(x) + c$



$(\Delta x, \Delta y)$

$\Delta x = x_1 - x_0$

$\Delta y = y_1 - y_0$

$\gamma(t) = (x(t), y(t))$

x, y are continuous on $[a, b]$

diff on (a, b)

$\gamma'(t) = (x'(t), y'(t))$

$\underline{(\Delta x, \Delta y)}$ is parallel to

$\underline{(x'(t), y'(t))}$

for some $t \in (a,b)$.

$$\Delta x\, y' - \Delta y\, x' = 0$$

$\to$ $(\Delta x, \Delta y)$ and (x', y') lie on the same line.

$(\Delta x, \Delta y) = \lambda (x', y')$ for some $\lambda \in \mathbb{R}$

$(x', y') = \lambda (\Delta x, \Delta y)$ — — — —

Generalized MVT

Suppose $f, g : [a,b] \to \mathbb{R}$ are continuous and that f, g are differentiable on (a,b).

Then there exists $c \in (a,b)$ such that

$$(f(b) - f(a))\, g'(c) - (g(b) - g(a))\, f'(c) = 0.$$

Moreover, if $g'(x) \neq 0$ for all $x \in (a,b)$

Then $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Pf: easy Cor of MVT

Homework. $\square$

Cor L'Hopital's Rule $\left(\frac{0}{0}\right)$

Suppose f and g are continuous on an interval $[a,b]$ containing a point c in (a,b). Suppose moreover that f, g are differentiable on $(a,b) \setminus \{c\}$.

Finally suppose $f(c) = g(c) = 0$ $\frac{f(x)}{g(x)}$

and that $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ for

some L. Then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$ as well.

Pf: Generalized MVT.

HW

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = \frac{1}{1} = 1$$

$$\frac{0}{0}$$

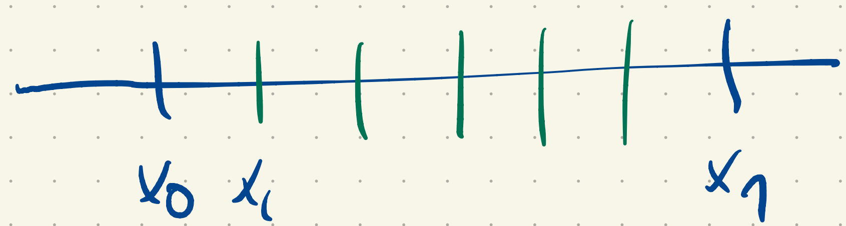
Integration.

$$\int_a^b f(x)\,dx$$

① divide $[a,b]$ into n equal width subintervals of width $\Delta x = \frac{b-a}{n}$

② Let $x_k = a + k\Delta x \qquad 0 \le k \le n$

$x_0, x_1, \dots, x_n$



③ In each interval pick a sample point

$x_k^* \in [x_{k-1}, x_k]$.

④ Form $\sum_{k=1}^{n} f(x_k^*) \Delta x := S_n$

⑤ $\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$.

ⓓ Do the intervals need to be equal width

(b) S_n depends on the choice of sample points.

Does the limit?

(c) Does S_n even converge?

Darboux

f, g

$$f''(x) = g''(x) \quad \forall x \in [a,b]$$

$$f'(x) = g'(x) + c$$

$$F = f - g$$