Convergence of sequences of functions

$(f_n), f : A \to \mathbb{R}$

$f_n \to f$ pointwise, if

for all $x \in A$, $f_n(x) \to f(x)$.

$f_n(x) = x^n$ on $[0,1]$

$f_n \to f$ pointwise

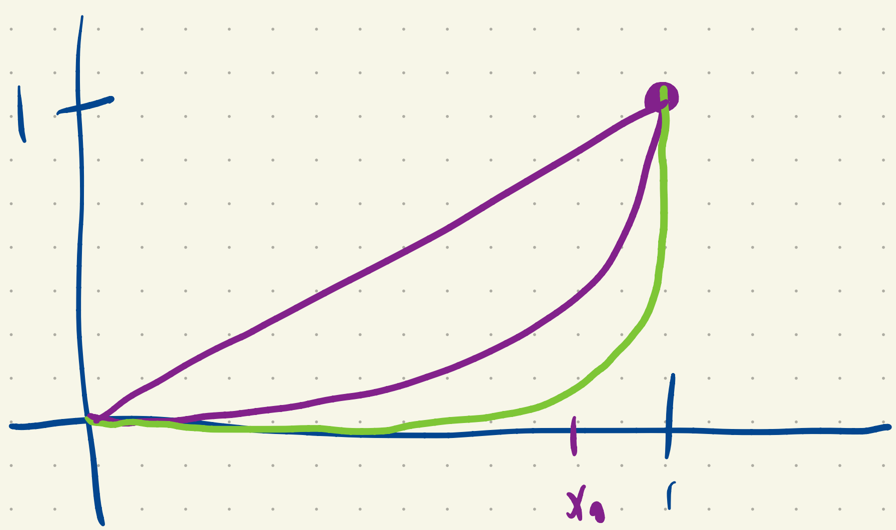
$$f_n(0) = 0 \quad \forall n \longrightarrow 0$$

$$f_n\left(\tfrac{1}{3}\right) = \left(\tfrac{1}{3}\right)^n \longrightarrow 0$$

$$f_n\left(\tfrac{2}{3}\right) = \left(\tfrac{2}{3}\right)^n \longrightarrow 0$$

$$f_n(1) = 1^n = 1 \quad \forall n$$

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$



$f_n \to f$ <u>uniformly</u> if

for every $\varepsilon > 0$ there exists N so

if $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ for

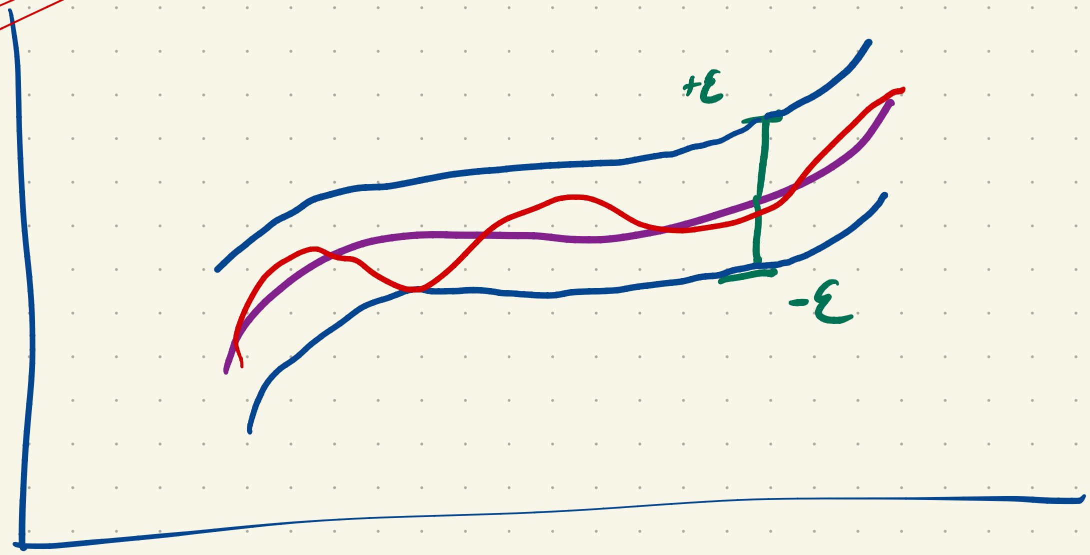
all $x \in A$.

Compare: $f_n \to f$ pointwise if

for all $x \in A$ and for all $\varepsilon > 0$ thee exists N

so if $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

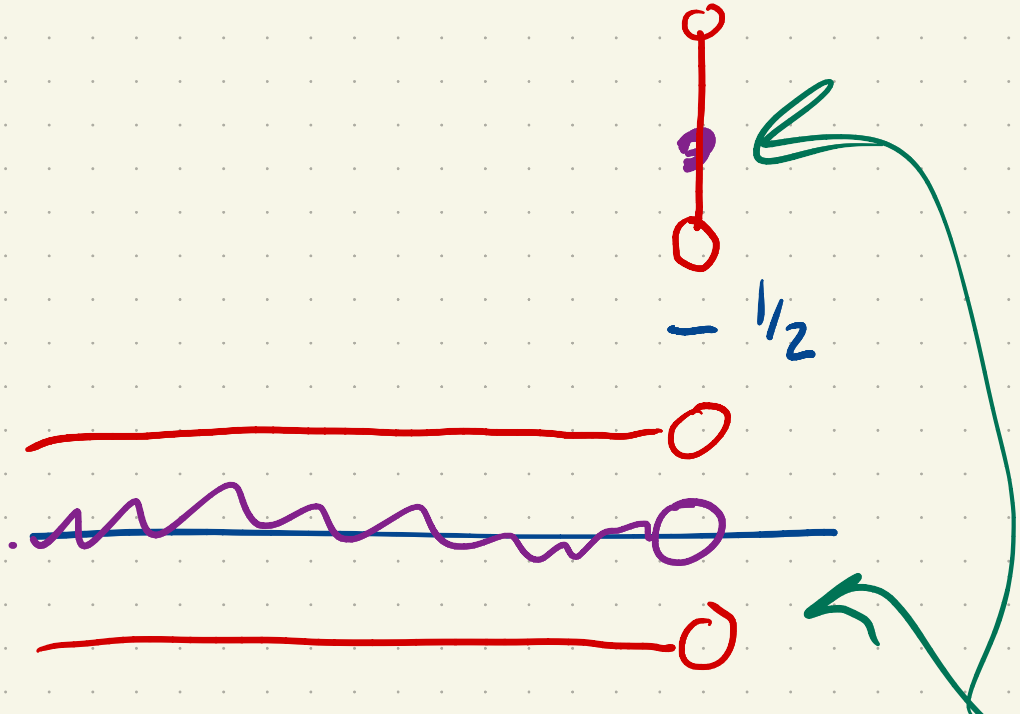
Uniform



N

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$$

$$\forall\, x \in A$$



$x^n = f_n(x)$

$\varepsilon = \frac{1}{4}$

N

$n \geq N$, sup $\leq$ n

Uniform convergence preserves continuity.

Prop. Suppose $(f_n), f : A \to \mathbb{R}$ and

$$f_n \rightrightarrows f.$$

unif. conv.

If for some $c \in A$ f_n is continuous at c for all n, then f is also continuous at c.

Pf: Let $\varepsilon > 0$. We can find N so if $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all

$x \in A$. Since f_N is continuous at c there exists $\delta > 0$ such that $|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$ for all $x \in A$ with $|x - c| < \delta$.

Then if $x \in A$ and $|x-c| < \delta$ then

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

(unif. conv.) (cont. of (unif. conv.)

$f_N)$

$= \varepsilon.$

$\square$

Does uniform conv. preserve differentiability? No

Does uniform conv. preserve Riemann integrability?

Yes

$$\int_a^b f_n \to \int_a^b f$$

"The limit of integrals is the integral of the limit"

$$f_n(x) = \sqrt{\left(\frac{1}{n}\right)^2 + x^2}$$

$$f_n \rightrightarrows |\cdot|$$

$$|x| \leq \sqrt{\left(\frac{1}{n}\right)^2 + x^2} \leq \frac{1}{n} + |x| \quad \forall x \in \mathbb{R}$$

$$\Updownarrow$$

$$x^2 \leq \left(\frac{1}{n}\right)^2 + x^2 \leq \overbrace{\left(\frac{1}{n} + |x|\right)^2}^{\frac{1}{n^2} + x^2 + \frac{2}{n}|x|}$$

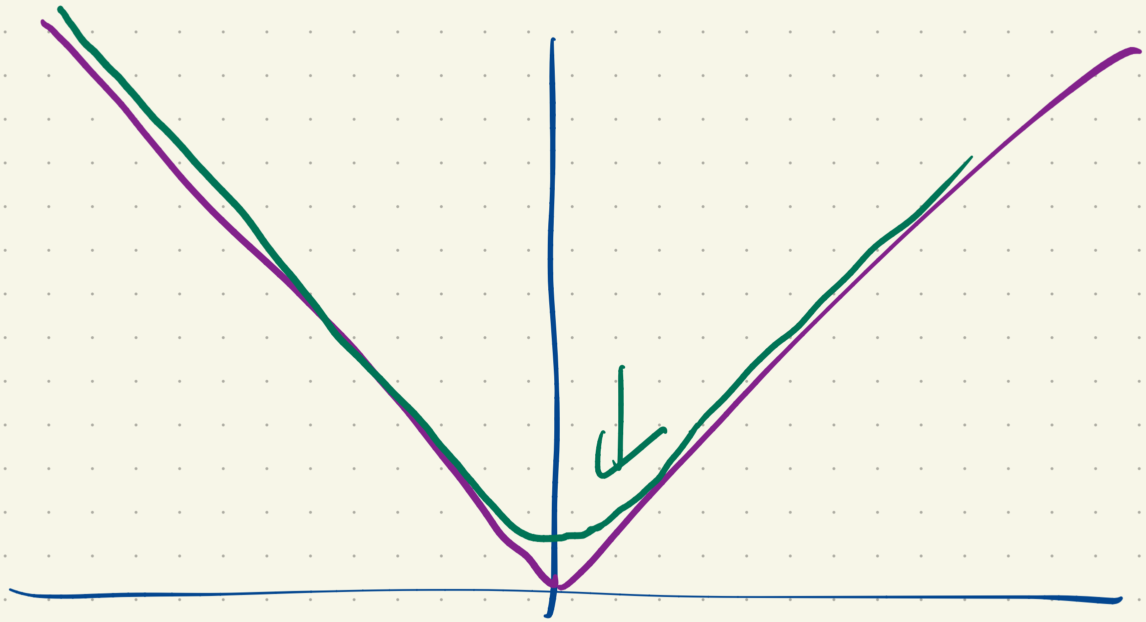
$$-\frac{1}{n} \leq 0 \leq \sqrt{\left(\frac{1}{n}\right)^2 + x^2} - |x| \leq \frac{1}{n}$$

$$\left| f_n(x) - |x| \right| \leq \frac{1}{n} \quad \forall x \in \mathbb{R}$$

$\varepsilon \quad \frac{1}{N} < \varepsilon.$ $\quad n \geq N, \quad \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

$N \quad \left| f_n(x) - |x| \right| < \varepsilon \quad \forall x \in \mathbb{R}$

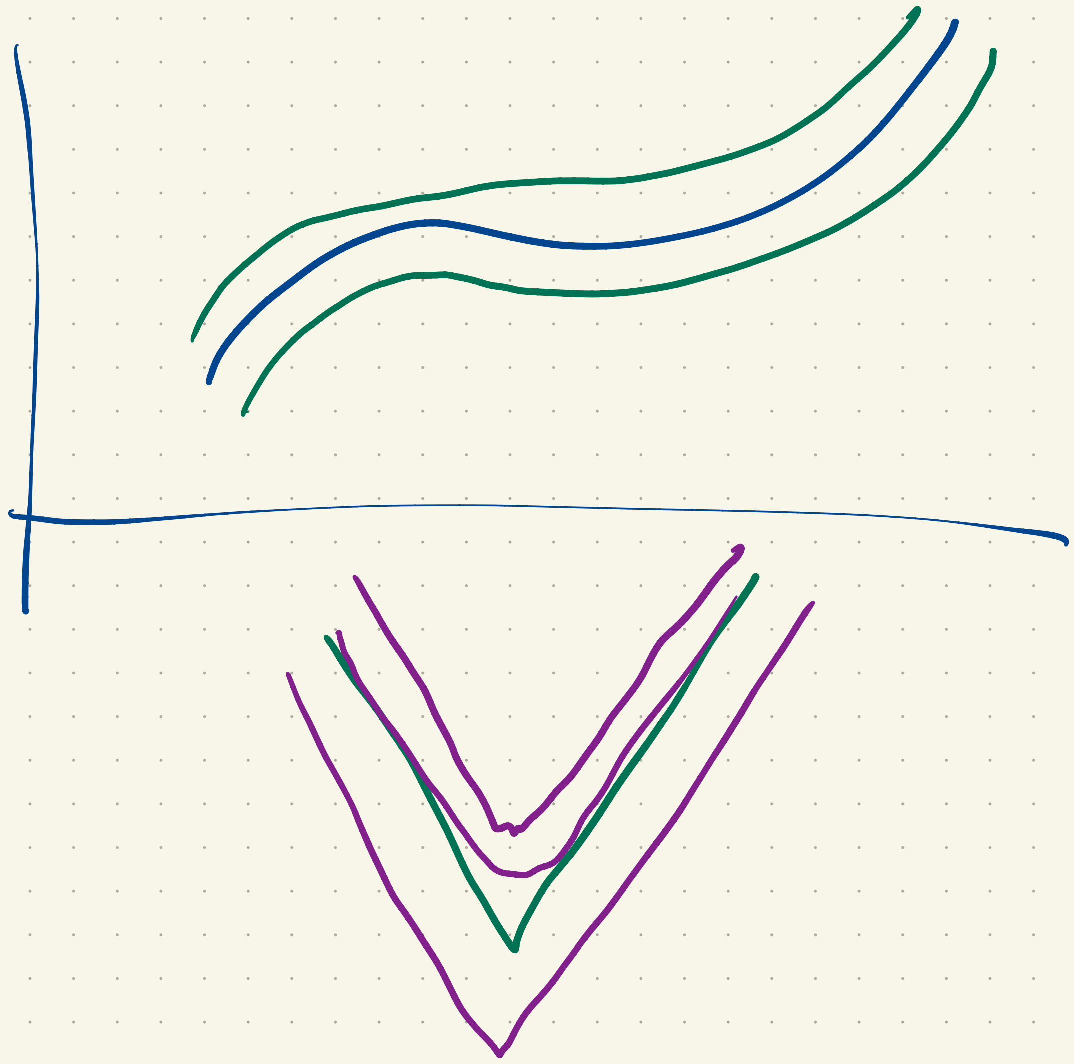
if $\quad n \geq N$



$\boxed{\begin{array}{l} g(x) = |x| \\ \text{is not diff} \\ \text{at } x = 0 \end{array}}$

$$f_n(x) = \sqrt{\left(\tfrac{1}{n}\right)^2 + x^2}$$

$$f(x) = |x| \qquad f_n \rightrightarrows f$$



In order to say something about derivatives we need to assume more.

Def: $(f_n) : A \to \mathbb{R}$ is uniformly Cauchy if for all $\varepsilon > 0$ there exists $N > 0$ such that

if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \varepsilon$.

(x_n) is Cauchy if for all $\varepsilon > 0$ there exists N

such that if $n, m \geq N$ $\underline{|x_n - x_m| < \varepsilon}$

Prop: A sequence of functions converges uniformly iff it is uniformly Cauchy.