Axiom of Completeness:

Every nonempty subset of $\mathbb{R}$ that is bounded above admits a supremum.

Consequences: ① $\mathbb{N}$ is not bounded above (in $\mathbb{R}$)

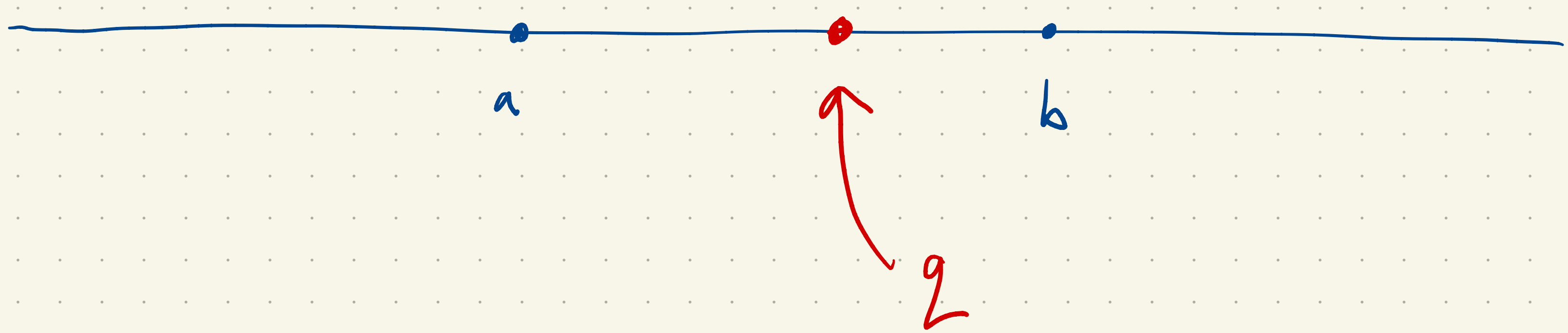
② Numbers of the form $1/n$, $n \in \mathbb{N}$ can be made as small as you please.

Axiom of $\mathbb{N}$

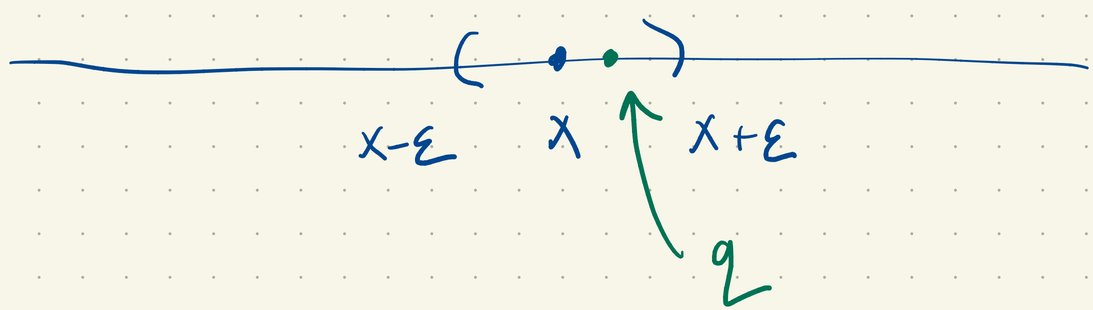
Well ordering: every nonempty subset of $\mathbb{N}$ has a least element

Prop: Suppose $a, b \in \mathbb{R}$, $a < b$. Then there exists $q \in \mathbb{Q}$ such that

$$a < q < b.$$



Known as the <u>density</u> of rational numbers.



Pf: We will assume that $a \geq 0$; the general case then follows by an easy argument.

Pick $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$ (see Prop 1.4.2 (iii)).

Let $S = \{ m \in \mathbb{N} : \frac{m}{n} > a \}$.

There exists $M \in \mathbb{N}$ such that $\underbrace{M > na}_{\frac{M}{n} > a}$ and hence $S \neq \emptyset$.

By the Well ordering of $\mathbb{N}$, S admits a least element, call it m.

I claim that $a < \frac{m}{n} < b$.

Indeed, $a < \frac{m}{n}$ by the definition of S.

Moreover $\frac{m-1}{n} \le a$ (either because $m = 1$ and hence $\frac{m-1}{n} = 0$ or because $m-1 \in \mathbb{N}$ and m is the least element of S).

$m-1 \notin S$

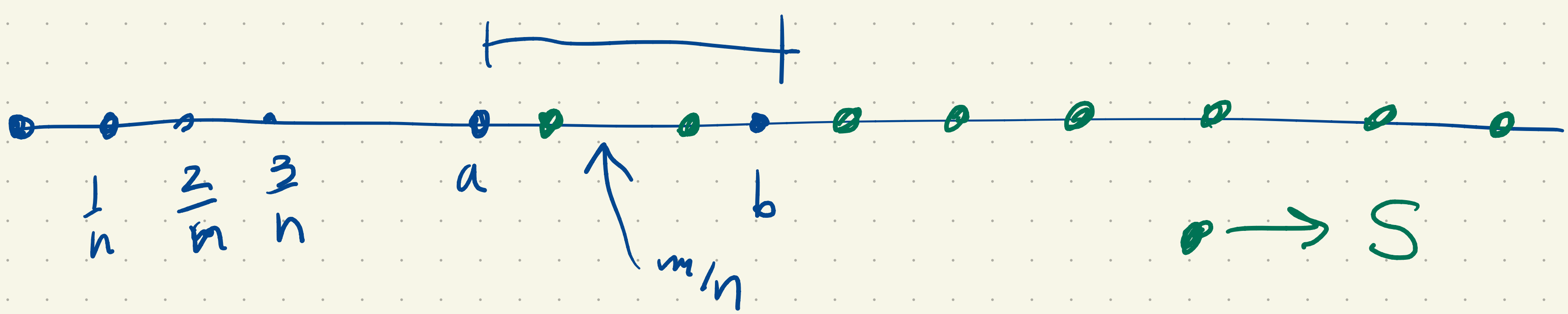
But then

$$\frac{m}{n} \le a + \frac{1}{n} < a + (b-a) = b.$$

$\square$

$b - a > \frac{1}{n}$

$1/n < b-a$



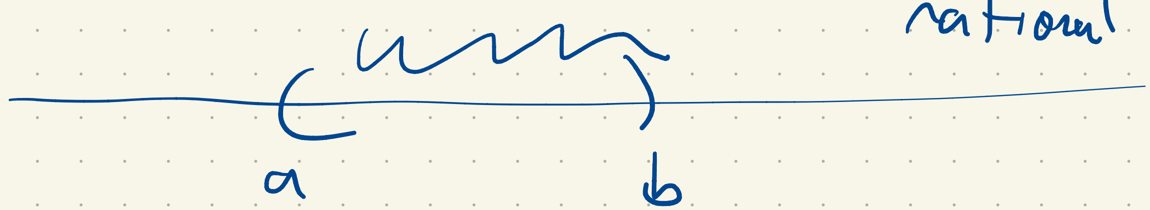
Def: If $x \in \mathbb{R}$ and $x \notin \mathbb{Q}$ then x is irrational.

We will see shortly that there is a real number satisfying $x^2 = 2$. We proved earlier that such a number cannot be rational and is hence irrational.

$\mathbb{R} \setminus \mathbb{Q}$

Next homework: You'll use the density of $\mathbb{Q}$ to prove the density of the irrational numbers.

There are both rational and irrational numbers.

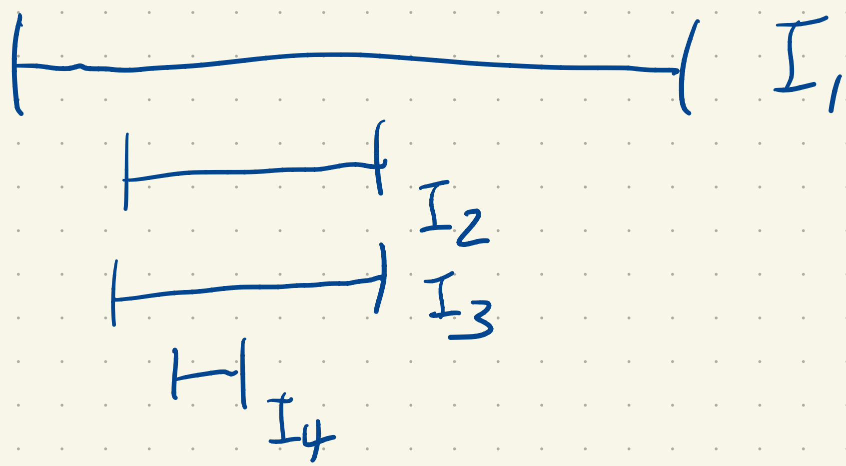


Nested Interval Property.

Idea: $\mathbb{R}$ has "no holes." Captures, in some sense, the word completeness.

Intervals $I_n = [a_n, b_n]$, $a_n \le b_n$

nested: $I_{n+1} \subseteq I_n$



$I_1 = [0,1]$

$I_2 = [0,1]$

$I_3 = [0,1]$

Claim: $\bigcap I_k \neq \emptyset$

$\bigcap I_k = [0,1]$

Requirements: the intervals have to be closed.

$$I_n = (0, \tfrac{1}{n}]$$

$$I_{n+1} \subseteq I_n \qquad \frac{1}{n+1} \leq \frac{1}{n}$$

$$\bigcap I_k = \emptyset$$

Suppose to the contrary that $x \in \bigcap I_k$.

So $x \in I_k$ for all k and hence $x > 0$.

But then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < x.$$

But then $x \notin I_n$, a contradiction

$$I_n = \{ z : 0 < z \leq \tfrac{1}{n} \}$$

The intervals have to be nested.

$I_k = [k-1, k]$

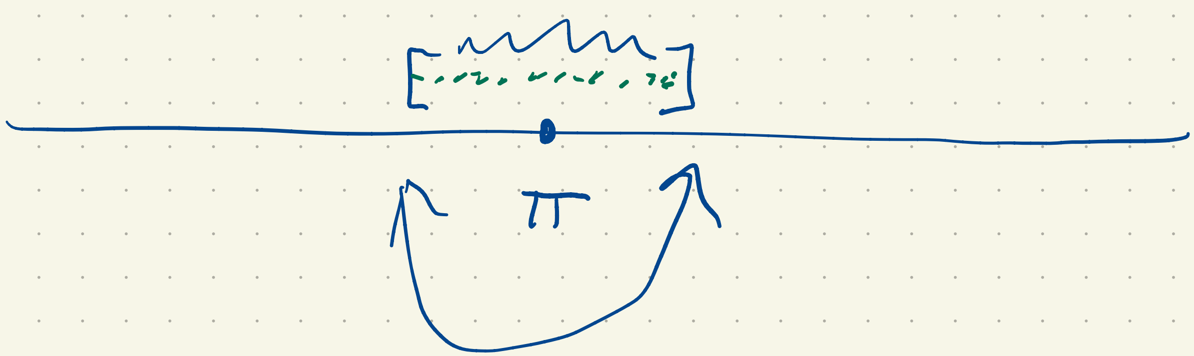
$I_1 \cap I_2 = \{1\}$

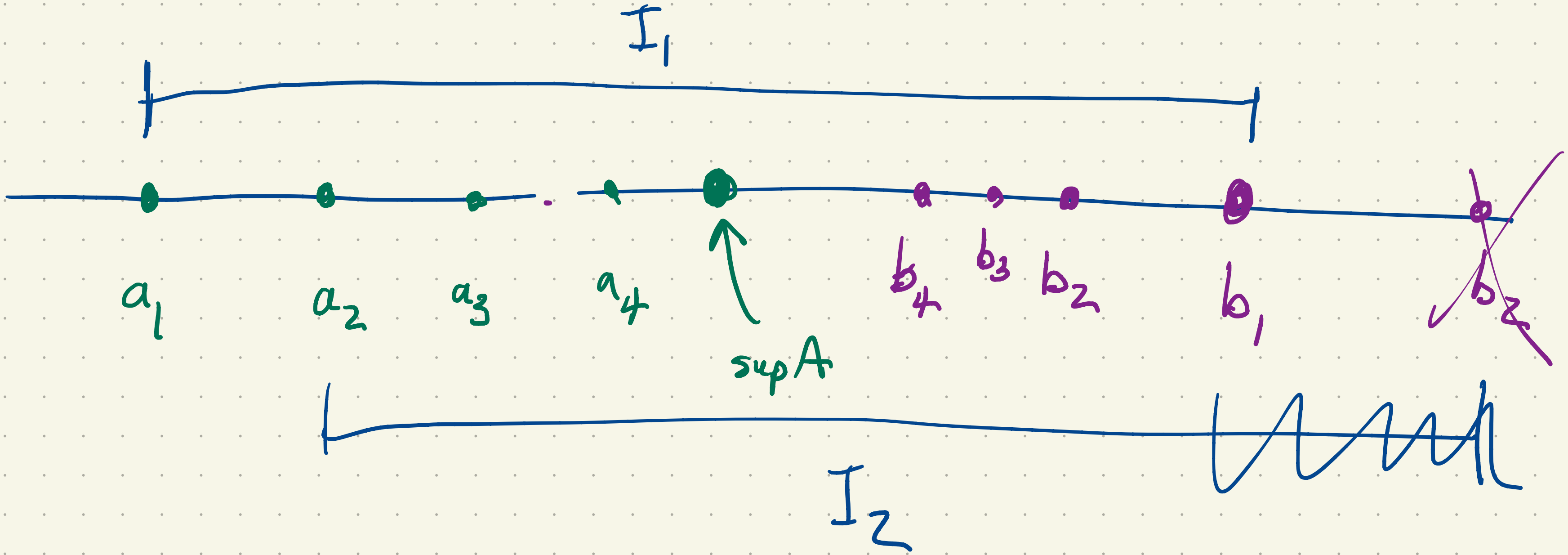
$\bigcap I_k$

$\underbrace{I_1 \cap I_2} \cap I_3 = \emptyset$

$I_3 = [2, 3]$

$\{q \in \mathbb{Q} : \pi - \frac{1}{n} \leq q \leq \pi + \frac{1}{n}\}$





$I_4 = [a_4, b_4]$

$a_k \le \sup A$ for all k's.

$\sup A \le b_k$ for all k

$A = \{ a_k \}$ nonempty, bounded above.

because each b_k is an upper bound

has a supremum!

$a_k \le \sup A \le b_k$ for all k.

$\sup A \in I_k$ for all k.

$\sup A \in \bigcap I_k$