**Last class:**

Introduced $\mathbb{R}$ as an ordered field.

1. Field
2. Order

$\mathbb{C}$, complex, field, but not ordered

$\mathbb{Q}$ : also an ordered field

3. Axiom of Completeness

Def: Let $A \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is an <u>upper bound</u> for $A$ if for all $a \in A$, $a \leq x$.

e.g. $A = [0,1]$ 7 is an upper bound!

12 is an - - - - - .

$A = \sin(\mathbb{R}) = \{\sin(x) : x \in \mathbb{R}\}$

1 is an upper bound

Def: A set $A \subseteq \mathbb{R}$ is <u>bounded above</u> if it admits an upper bound.

e.g. $A = [0,1]$ is bounded above

Is $\mathbb{N} \subseteq \mathbb{R}$ bounded above?

No largest natural number, true.

$n \in \mathbb{N}$, look at $n+1$.

How about $\emptyset$?

$\emptyset$ is bounded above means:

There exists $x \in \mathbb{R}$, such that for all $a \in \emptyset$, $a \leq x$.

$\emptyset$ is not bounded above means

for all $x \in \mathbb{R}$ there exists $a \in \emptyset$ such that $a > x$

Exercise: Every $x \in \mathbb{R}$ is an upper bound for $\emptyset$.

Def: Let $A \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is a <u>supremum</u> or <u>least upper bound</u> for $A$ if

① $x$ is an upper bound for $A$

② Whenever $y$ is some upper bound for $A$, $x \leq y$.

We write $x = \sup A$ if $x$ is a supremum for $A$.

↳ \sup

② encodes leastness

E.g. $A = \{0, 1\}$ 1 is a supremum for $A$

① $1 \geq 0$? ✓

$1 \geq 1$? ✓ 1 is an upper bound.

② Verify leastness.

Let $y$ be some upper bound for $A$.

Since it is an upper bound, in particular,

$y \geq 1$.

Theorem: Suprema are unique.

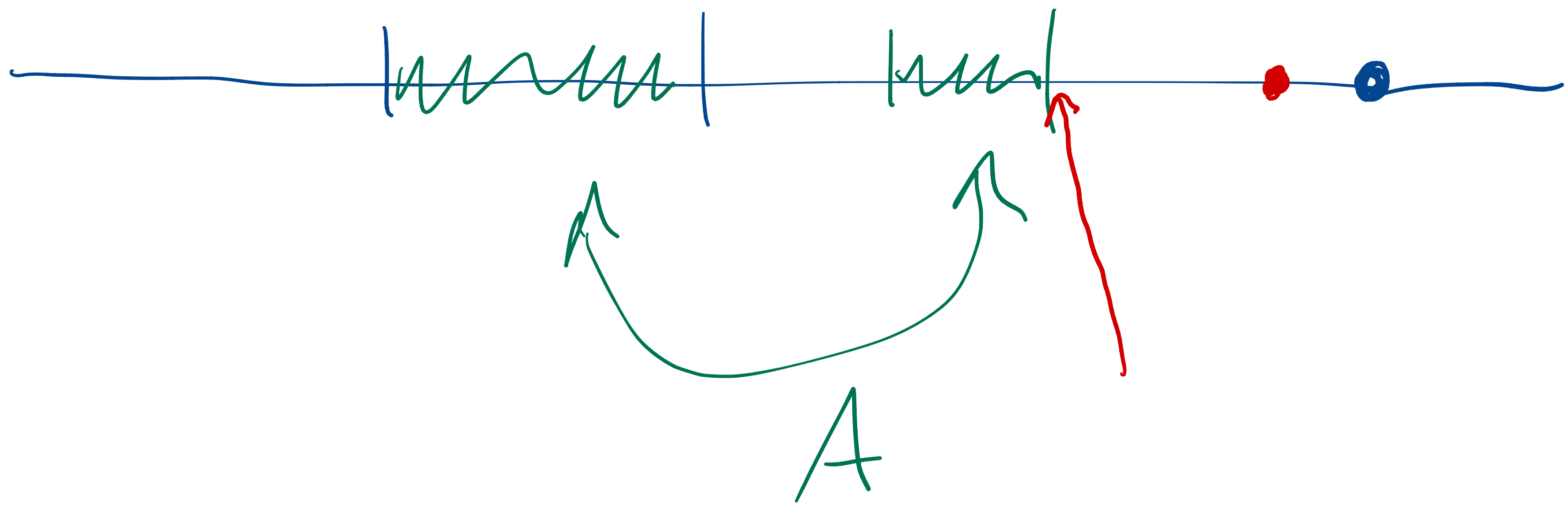
Pf: Let $A \subseteq \mathbb{R}$ and suppose $x, y$ are suprema for $A$.

In particular $x$ is an upper bound.

Since $y$ is a supremum for $A$, $y \leq x$.

The same reasoning shows $x \leq y$. Hence $x = y$.

$\square$


A