Numbers:

$\mathbb{N}$ $1, 2, 3, 4, \ldots$

operation: $+, \cdot$ induction!

$\mathbb{N}_0$ $0, 1, 2, 3, \ldots$

integers $\mathbb{Z}$ $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

operations: all of $\mathbb{N}$ (no induction)

two special ones: 0 $0 + a = a$

1 $1 \cdot a = a$

rational $\mathbb{Q}$

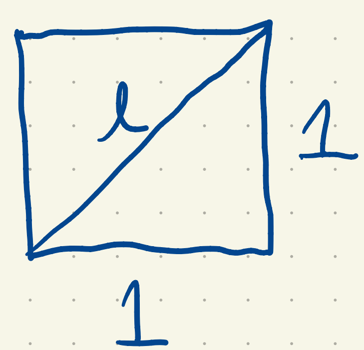
intuitively: a/b $\quad a, b \in \mathbb{Z}$

$b \neq 0$.

$$\frac{a}{b} = \frac{a'}{b'} \quad \text{if} \quad ab' = a'b$$

Operations: all of $\mathbb{Z}$, plus division

There are lengths that are not rational



$$1^2 + 1^2 = l^2$$

$$l^2 = 2$$

Theorem: There does not exist a rational number x such that $x^2 = 2$.

Pf: Suppose to the contrary that $q \in \mathbb{Q}$ and $q^2 = 2$.

We can write $q = a/b$ where $a, b \in \mathbb{Z}$, $b \neq 0$ and a and b have no common factors.

Notice $a = qb$ and

$$a^2 = q^2 b^2 = 2b^2.$$

Hence a^2 is even, as is a. Thus $a = 2c$ for some $c \in \mathbb{Z}$. But then $a = qb$ so

$$2c = qb$$

and

$$4c^2 = a^2 b^2$$
$$= 2b^2.$$

But then $b^2 = 2c^2$ and b is even by the same reasoning as before. Hence a and b have 2 as a common factor, a contradiction. $\square$

That leads us to $\mathbb{R}$, the real numbers.

Rules: ① Field

a) Has commutating associative binary operations

$+, \cdot$.

b) Has additive, mult. identities $(0, 1)$

c) Has additive inverse $a + (-a) = 0$

d) Has mult - - . . $a \cdot a^{-1} = 1$

$(a \neq 0)$

c) Distributive law $a(b+c) = ab + ac$.

② Ordered. Notion of $<$. If $a, b \in \mathbb{R}$
$a \neq b$, then $a < b$ or $b < a$.