The aim of this worksheet is to prove the following theorem.

Theorem 1 (Alternating Series Theorem)**.** *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a monotone decreasing sequence of non-negative numbers such that* $\lim_{n\to\infty} a_n = 0$. *Then the series*

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converges.

In the following, we suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ satisfiying $a_1 \geq a_2 \geq a_3 \cdots > 0$ and $\lim_{n\to\infty} a_n = 0$. Let s_n denote the n^{th} partial sum of the series.

1. Show that the subsequence s_{2j+1} is a decreasing sequence.

2. Show that the subsequence s_{2j} is an increasing sequence.

3. Use the equation $s_{2j+1} = s_{2j} + a_{2j+1}$ (and the previous two problems) to show that $\{s_{2j}\}_{j=1}^{\infty}$ is bounded above by s_1.

4. Find (with proof) a lower bound for s_{2j+1}.

5. Conclude that there are numbers L and L' such that $\lim_{j\to\infty} s_{2j} = L$ and $\lim_{j\to\infty} s_{2j+1} = L$.

6. Show that in fact $L = L'$.

7. Look at problem 2.3.5 in your text. Use it to conclude the proof. [You should be motivated now to prove problem 2.3.5!]

If you are bored, try the following alternative approach to proving the theorem.

8. Let $j \in \mathbb{N}$. Suppose $k \geq 2j-1$. Show $s_{2j} \leq s_k \leq s_{2j-1}$.

9. Conclude that if $n > m \geq 2j-1$ then

$$|s_n - s_m| \leq a_{2j}.$$

10. Deduce that the sequence of partial sums is Cauchy and therefore converges to a limit.