Please see the rules on the second page.

1. Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in X$ and that $\lim_{x\to c} f(x)$ exists. Show that the limit is non-negative. Provide two proofs, one $\epsilon - \delta$ style, and the other using the sequential characterization of limits.

2. Let a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges. Suppose that $|x| < M$. Show that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. Give an example to show that divergence is possible if $|x| = |M|$. Hint: $(a_n M^n)$ converges to zero, and is hence bounded.

3. Suppose $f : (0,1] \to \mathbb{R}$ is uniformly continuous. Show that $\lim_{x\to 0} f(x)$ exists.

4. Abbott 4.3.11

5. Suppose that $f : (0,1) \to \mathbb{R}$ is continuous and that $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 1} f(x) = \infty$. Show that f obtains a minimum on $(0,1)$.

6. Show that if $f : [a,b] \to \mathbb{R}$ is strictly increasing and continuous, then it has a continuous inverse function $f^{-1} : [f(a), f(b)] \to [a,b]$. Use this result to show that $x^{1/n}$ is continuous for each $n \in \mathbb{N}$. You may use any homework problems you have done to help with this. But your proof must give a careful demonstration that the domain of f^{-1} is the whole interval $[f(a), f(b)]$, that the image is exactly $[a,b]$, and that f^{-1} is increasing.

7. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and that $f([0,1]) \subseteq [0,1]$. Prove that there is a solution of the equation $f(x) = x$.

8. Abbott 5.2.12

Hint: The hard part about this problem is showing that f^{-1} has a derivative at all. In my mind, the easiest approach is to use the sequential characterization of limits. That is, given $c \in [f(a), f(b)]$, let y_n be a sequence in the interval converging to c. Now see if you can compute

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(c)}{y_n - c}. \tag{1}$$

If you get the same answer L regardless of what sequence y_n you pick, then

$$\lim_{y\to c} \frac{f^{-1}(y) - f^{-1}(c)}{y - c} = L.$$

You will likely find it useful to rephrase the limit in (1) in terms of f rather than f^{-1}.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.

- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You my not discuss the exam with anyone else until after the due date/time.
- You may freely use any result proved in class, or in your text, or on a homework assignment, but you must cite any such result. However, you may not cite a result that makes a problem trivial. Ask me if you have any questions about using a particular result.
- You are permitted to use other real analysis texts besides the course text in writing up your solutions, but you must acknowledge any such use.
- No other sources or aids are permitted.
- Each problem is weighted equally.
- The due date/time is absolutely firm.