Please see the rules on the second page.

1. Let A and B be nonempty sets that are bounded above. Suppose $\sup A < \sup B$. Prove that there is an element of B that is an upper bound for A.

2. In class we proved that $\mathbb{N}^2$ is countably infinite. Use this fact and a proof by induction to show that $\mathbb{N}^n$ is countably infinite for every $n \in \mathbb{N}$.

3. Compute
$$\lim_{n\to\infty} \frac{3^n}{n!}.$$
A fully rigorous proof will likely involve at some point a proof by induction.

4. Text: 2.3.5

5. Suppose $\mathcal{F}$ is a collection of open intervals such that if $I, J \in \mathcal{F}$ and $I \neq J$, then $I \cap J = \emptyset$. Prove that $\mathcal{F}$ is countable.

6. Let (x_n) be a sequence converging to L. Define
$$y_n = \frac{x_1 + \cdots + x_n}{n}.$$
That is, y_n is the average of the first n terms of the sequence of x_n. Show that $\lim y_n = L$ as well.

7. Use the Bolzano-Weierstrass theorem to prove the Monotone Convergence Theorem without assuming any other form of the Axiom of Completeness.

8. Suppose (x_n) is a sequence and that for all $n \geq 2$,
$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|.$$
Show that the sequence converges.

9. Let (a_n) and (b_n) be sequences with $b_n \geq 0$ for all n and $\lim_n b_n = 0$. We say that $a_n = \mathcal{O}(b_n)$ if there is a constant C such that $|a_n| \leq C b_n$ for all n. Roughly speaking, $a_n = O(b_n)$ if the sequence a_n converges to zero at least as fast as the sequence b_n.

   Suppose a_n and b_n are sequences with $b_n > 0$. Suppose also that $\lim_n \frac{a_n}{b_n} = L$ for some number L. Prove that $a_n = \mathcal{O}(b_n)$.

10. Suppose (a_n) and (b_n) are sequences with $b_n \geq 0$ and $a_n = \mathcal{O}(b_n)$.

    a) Suppose that $\sum b_n$ converges. Prove that $\sum a_n$ converges also.

    b) Suppose that $\sum a_n$ diverges. Prove that $\sum b_n$ diverges.

    c) Determine if $\sum_{n=1}^{\infty} \sqrt{\frac{n^3-3n+2}{8n^4+n^2+22}}$ converges.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You my not discuss the exam with anyone else until after the due date/time.
- You may freely use any result proved in class, or in your text, or on a homework assignment, but you must cite any such result. However, you may not cite a result that makes a problem trivial. Ask me if you have any questions about using a particular result.
- You are permitted to use other real analysis texts besides the course text in writing up your solutions, but you must acknowledge any such use.
- Each problem is weighted equally.
- The due date/time is absolutely firm.