Please see the rules on the second page.

1. Suppose (x_n) and (y_n) are sequences such that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = \infty$. Show that $\lim_{n\to\infty} x_n/y_n = 0$.

2. A number is *algebraic* if it is a solution of a polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where each a_k is an integer, $n \geq 1$, and $a_n \neq 0$. Show that the collection of all algebraic numbers is countable.

3. Let p be a fifth order polynomial, so $p(x) = \sum_{k=0}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Prove that there is a solution of $p(x) = 0$.

4. Let $\sum_{k=1}^{\infty} a_k$ be a series. Suppose moreover that $\lim_{k\to\infty} |a_k|^{\frac{1}{k}}$ exists and equals L. Show that the series converges absolutely if $L < 1$ and diverges if $L > 1$.

5. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there is a number L such that $f(x) = f(x+L)$ for all $x \in \mathbb{R}$. Show that a continuous, periodic function is uniformly continuous.

6. Use the Nested Interval Property to deduce the Axiom of Completeness without using any other form of the Axiom of Completeness. *Hint:* Look at the proof of the Bolzano-Weierstrass Theorem.

7. Let (r_n) be an enumeration of the rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/n & x = r_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Determine, with proof, where f is continuous.

8. Abbott 5.2.6

9. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sin(kx).$$

Show that f is differentiable.

10. Suppose that $f : [0,1] \to \mathbb{R}$ is twice differentiable, $f(0) > 0$, $f(1) = 1$, and $f'(1) < 1$. Suppose also that $f'' > 0$ on $[0,1]$. Show that there does not exist a solution of the equation $f(x) = x$ in $[0,1)$.

11. 7.2.5

12. 7.5.8(a-d)

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You may not discuss the exam with anyone else until after the due date/time.
- You may freely use any result proved in class, or in your text, or on a homework assignment, but you must cite any such result. However, you may not cite a result that makes a problem trivial. Ask me if you have any questions about using a particular result.
- You are permitted to use other real analysis texts besides the course text in writing up your solutions, but you must acknowledge any such use.
- Each problem is weighted equally.
- The due date/time is absolutely firm.