Last class:

$$d : \Lambda^1 \to \Lambda^2$$

$$(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i$$

$$\mathcal{A}(R, S) = \begin{bmatrix} 0 & R_1 & R_2 & R_3 \\ & 0 & S_3 & -S_2 \\ & & 0 & S_1 \\ & & & \end{bmatrix}$$

$$d\omega = \mathcal{A}(\partial_0 \vec{\omega} - \nabla \omega_0, \nabla \times \vec{\omega})$$

$$* : \Lambda^2 \to \Lambda^2 \qquad \mathcal{A}(-S, R)$$

$$*d : \Lambda^2 \to \Lambda^1 \qquad [\operatorname{div} S, -\nabla \times R + \partial_0 S]$$

$$\delta = *d* : \Lambda^2 \to \Lambda^1$$

$$\mathcal{M} = -\delta d$$

For a stationary charge distribution

$$\mathcal{M}\omega = \frac{1}{c\varepsilon_0}[c\rho, -j] \quad \text{in any coord system}$$

where $\omega = [\phi, 0]$ $\qquad -\Delta \phi = \dfrac{1}{4\pi\varepsilon_0}\rho$

Given a current density $\begin{bmatrix} c\rho \\ \vec{j} \end{bmatrix}$ on spacetime,

a solution of Maxwell's equations is a 1-form ω

satisfying

$$\mathcal{M}\omega = \frac{1}{c\varepsilon_0} j \qquad j = [c\rho, -\vec{j}]$$

The associated EM field is $d\omega$

We define E, B by

$$\mathcal{F}(E, -cB) = d\omega.$$

On your HW: $\operatorname{div} E = \frac{1}{\varepsilon_0}\rho$ Gauss' Law

$$c\nabla \times B + \partial_0 E = \frac{1}{c\varepsilon_0} \vec{j}$$ Ampere's Law

are a consequence of $\delta\, d\omega = 0$.

But also I mentioned $d^2 = 0$ always.

$$\begin{tikzcd}
\Lambda^0 \arrow[r, "d"] \arrow[rr, bend right, "\text{you'll verify}"'] & \Lambda^1 \arrow[r, "d"] & \Lambda^2 \arrow[r] & \Lambda^3
\end{tikzcd}$$

$d^2 : \Lambda^1 \to \Lambda^3$ is hard since I didn't tell you what Λ^3 is.

But $* d d : \Lambda^1 \to \Lambda^1$ you can check.

Exercise (HW) $* d d = 0$.

As a consequence, we obtain two more equations:

$$\operatorname{div} B = 0 \qquad \text{No mag. sources}$$

$$c\,\partial_0 B + \nabla \times E = 0 \qquad \text{Faraday's Law}$$

which are nothing more than a reflection of $d^2 = 0$.

$d\omega = \mathcal{F}(E, -cB)$ by def of E, B.

$*d\omega = \mathcal{F}(-cB, -E)$

$*d*d\omega = (-\operatorname{div} E,\ c\nabla\times B - \partial_0 S)$

$-*d*d\omega = (\operatorname{div} E,\ -c\nabla\times B + \partial_0 E)$

$*d\ \mathcal{F}(R,S) = [\operatorname{div} S,\ -\nabla\times R + \partial_0 S]$

So ① $\operatorname{div} E = \frac{1}{\varepsilon_0}\rho$ (Gauss' Law)

$-c\nabla\times B + \partial_0 E = -\frac{1}{c\varepsilon_0} j$

② $\nabla\times B = \underbrace{\boxed{\frac{1}{c^2\varepsilon_0}}}_{\mu_0} j - \underbrace{\frac{1}{c}\partial_0}_{\frac{1}{c^2}\partial_t} E$ (Ampere's equation with Maxwell's addition)

Exercise: $*d\,d\omega = 0$ (another face of $d^2=0$)

$*d\ \mathcal{F}(E,-cB) = [-c\operatorname{div} B,\ -\nabla\times E + \partial_0(-cB)]$

③ $\operatorname{div} B = 0$ (no magnetic sources)

④ $\partial_t B + \nabla\times E = 0$ (Faraday's Law of Induction]

$\delta^2 = 0$ always as well

$$* d * \; * d * = \pm * \underbrace{d^2}_{=0.} *$$

You'll verify $\delta^2 : \Lambda^2 \to \Lambda^0 = 0$

$\delta : \Lambda^2 \to \Lambda^1$ by my recipe

$\delta : \Lambda^1 \to \Lambda^0$ $\quad \partial_0 \omega_0 - \operatorname{div} \vec{\omega}$.

As a consequence, there is a compatibility condition on the current:

$$-\delta d\omega = j$$

$$0 = -\delta^2 d\omega = \delta j$$

Which is exactly that $\operatorname{Div} J = 0$ $\quad J = \begin{bmatrix} c\rho \\ \vec{j} \end{bmatrix}$

which is conservation of charge density for all observers.

One more feature: gauge freedom.

Suppose ω is a solution of Maxwell's equations.

Let f be any function.

Then $\omega + df$ is again a solution of Maxwell's equations

$$-\delta d(\omega + df) = -\delta d\omega + -\delta d^2 f = -\delta d\omega = j$$

These are two faces of the same solution and correspond to a kind of coordinate change I don't have time to elaborate on.

Exercise: $-\delta d\omega = \Box \omega + d\delta\omega$

where $\Box \omega = (\Box \omega_0, \Box \omega_1 \dots)$

Suppose we can find an f such that $\delta d f = -\delta\omega$ at each time.

Then $\hat{\omega}$ is again a solution of Maxwell's equations

$\hat{\omega} = \omega + df$ and

$$\mathcal{M}\hat{\omega} = \Box\hat{\omega} + d\delta(\omega + df) = \Box\hat{\omega}$$

I.e. Maxwell's equations reduce to

$$\Box \hat{\omega} = j$$

an in homogeneous wave equation.

$\hat{\omega}$ at $t=0$
$\partial_t \hat{\omega}$ at $t=0$ yields a unique sol. (we saw only homog. version earlier)

Is this always possible?

$$\delta d f = -\delta \omega$$

$$\Box f = -\delta \omega$$ yes! just solve the wave eq with your choice of initial conds

If a solution ω of maxwell's eq's exists, then there also exists a solution $\hat{\omega}$ with $\delta \hat{\omega} = 0$.

We call such a solution a sol in Lorentz gauge.

Harder (TH final)

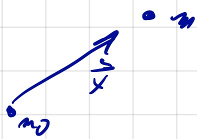
We can use this observation to solve maxwell's equations by solving wave equations.

Real subtleties.

Newton's Law of Gravity

Not old G

$$F = -\frac{G\, m_0\, m}{|x|^2} \frac{x}{|x|}$$



$$G_f = -\frac{G}{|x|^2}\frac{x}{|x|}$$

$$\Phi_f = \frac{G}{|x|}$$

$$\phi(x) = \int \Phi_f(x-y)\, \rho(y)\, dy$$

$$\nabla \phi = \int G_f(x-y)\, \rho(y)\, dy = \tilde{g}$$, grav field.

$$\Delta \phi = 4\pi \rho$$

For a particle of mass m

$$\frac{d}{dt} p = \vec{g}\, m$$

Two masses

$$\frac{d}{dt} m_I v = m \vec{g}$$

inertial mass, the one from Newton 2 in every context

grav. mass, only involved in gravity.

$$\frac{d}{dt} \vec{v} = \left(\frac{m}{m_I}\right) \vec{g}$$

Observation: For all matter the ratio $\dfrac{m}{m_I}$ is constant, and we can pick units so $m = m_I$

This is known as the "weak eq principle"

holds to one part in 10^{-13} at least.

So $\vec{a} = \vec{g}$ and all particles undergo the same acceleration in a fixed grav field.

Note: In electrostatics, there are neutral particles which, given no other forces, would travel in a straight line. So we can use them to help identify inertial frames.

No such joy for gravity.

We cannot measure the absolute size of the grav field.

$\left(\frac{\epsilon_0}{2} \left[|E|^2 + c^2 |B|^2 \right] \text{ for EM field.} \right)$

$\mu_0 \epsilon_0 = c^2$

$$\vec{g}(x)$$

$$\frac{d^2\alpha}{dt^2} = \vec{g}(\alpha(t))$$

Now add a monster $\vec{G} = -G e_3$

New path $\quad \frac{d^2\beta}{dt^2} = \vec{g}(\beta(t)) - G e_3$

But introduce coords $\hat{z} = z + \frac{1}{2} t^2 G$.

$$\hat{\beta}(t) = \beta(t) + \frac{1}{2} G t^2 e_3$$

$$\begin{aligned}
\hat{\beta}'' &= \beta'' + G e_3 \\
&= \vec{g}(\beta(t)) - G e_3 + G e_3 \\
&= \vec{g}\left(\beta(t) + \tfrac{1}{2} G t^2 e_3 - \tfrac{1}{2} G t^2 e_3\right) \\
&= g\left(\hat{\beta}(t) - \tfrac{1}{2} G t^2 e_3\right) = \hat{g}(\hat{\beta}(t))
\end{aligned}$$