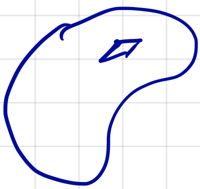
Last class: EM tensor

$$F = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & cB_3 & -cB_2 \\ -E_2 & -cB_3 & 0 & cB_1 \\ -E_3 & cB_2 & -cB_1 & 0 \end{bmatrix}$$

$$\frac{dp}{d\tau} = q\,G F V$$

$$L^t \hat{F} L = F$$

$$\underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{y}) = X^T F Y \qquad \underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{y}) = -\underset{\sim}{F}(\underset{\sim}{y}, \underset{\sim}{x})$$



To this point we have not used the relation

$$\vec{E} = \int_{\mathbb{R}^3} \vec{E}_\delta(x-y)\,\rho(y)\,dy \qquad \vec{E}_\delta = \frac{1}{4\pi\varepsilon_0}\frac{1}{|x|^2}\frac{x}{|x|}$$

Next goal: "derive" Maxwell's equations.

Notation:

Λ^0 functions on spacetime

Λ^1 covector fields on spacetime ($\omega(x) = [\omega_0(x), \dots, \omega_3(x)]$)

$$d : \Lambda^0 \longrightarrow \Lambda^1$$

In coords, $df = [\partial_0 f, \dots, \partial_3 f]$

It has a companion

$$\delta : \Lambda^1 \to \Lambda^0$$

$$\delta\omega = \partial_0\omega_0 - \partial_1\omega_1 - \partial_2\omega_2 - \partial_3\omega_3$$

$$\omega \longrightarrow V \longrightarrow \mathrm{Div}\,V = \delta\omega$$

$$\begin{tikzcd} & \underset{\sim}{\delta}\,\underset{\sim}{\omega} \arrow[dl] \arrow[dr] & \\ \hat{\delta}\hat{\omega} & & \delta\omega \end{tikzcd}$$

$$\hat{\delta}\hat{\omega}(Lx) = \delta\omega(Lx)$$

Exercise: $\hat{\delta}\hat{\omega} = \delta\omega$

$$\delta d f = \Box f$$

Λ^1: at each x is a map $\omega : V \to \mathbb{R}$.

Λ^2: at each x is a map $F: V \times V \to \mathbb{R}$,

bilinear, $F(V,W) = -F(W,V)$.

(e.g. the E-M field). We can rep. by an antisymmetric matrix.

Λ^3, Λ^4 as well.

And each of these is meant to be integrated

(Λ^1 over lines, Λ^2 over 2-surfaces, etc).

$\int_a^b \eta(\alpha'(s))ds$ is independent of param, but depends on direction.

Big picture

No metric needed

$$\begin{tikzcd} \Lambda^0 \arrow[r, shift left, "d"] & \Lambda^1 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Lambda^2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Lambda^3 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Lambda^4 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

uses the metric.

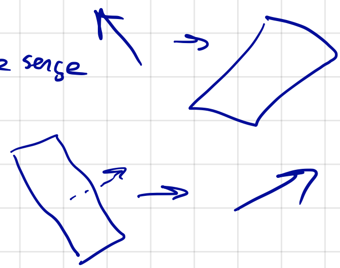
we've seen this

Moreover $d^2 = 0$

we'll visit this for Maxwell

Hodge $*$ $\quad * : \Lambda^i \longrightarrow \Lambda^{4-i}$

In some sense



$$\begin{aligned} * :\ & \Lambda^0 \to \Lambda^4 \\ :\ & \Lambda^1 \to \Lambda^3 \\ :\ & \Lambda^2 \to \Lambda^2 \\ :\ & \Lambda^3 \to \Lambda^1 \\ :\ & \Lambda^4 \to \Lambda^0 \end{aligned}$$

$\delta = * d *$ $\qquad$ e.g. $\Lambda^2 \to \Lambda^2 \to \Lambda^3 \to \Lambda^1$

(so $\delta^2 = 0$) $\qquad \Lambda^1 \to \Lambda^3 \to \Lambda^4 \to \Lambda^0$

But I'm going to try to avoid discussing Λ^3, Λ^4
($\Lambda^0, \Lambda^1, \Lambda^2$ can be rep in terms of matrices, vectors)

$$\Lambda^0 \underset{\delta}{\overset{d}{\rightleftarrows}} \Lambda^1 \qquad \delta d = \Box$$

$$\Lambda^1 \underset{\delta}{\overset{d}{\rightleftarrows}} \Lambda^2 \qquad -\delta d = \mathcal{M} \to \text{maxwell operator.}$$

So who is d, ∂?

$$\omega = [\omega_0, \omega_1, \omega_2, \omega_3]$$

$$(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i$$

Exercise $L^t \hat{d\omega} L = d\omega$ ($d\omega$ transforms like a 2-form)

Exercise $d^2 : \Lambda^0 \to \Lambda^2 = 0$

This is a deep generalization of $\nabla \times (\nabla f) = 0$

$$\operatorname{div}(\nabla \times V) = 0$$

To describe S I need some notation.

An antisymmetric 4x4 matrix has 6 independent entries.

Given $\quad R = [R_1, R_2, R_3]$

$$S = [S_1, S_2, S_3]$$

$$\mathcal{A}(R, S) = \begin{bmatrix} 0 & R_1 & R_2 & R_3 \\ -R_1 & 0 & S_3 & -S_2 \\ -R_2 & -S_3 & 0 & S_1 \\ -R_3 & S_2 & -S_1 & 0 \end{bmatrix}$$

Gives us a way to talk about them.

e.g. $\quad F = \mathcal{A}(E, -cB)$

Exercise: if $\quad \omega = [\omega_0, \vec{\omega}]$

$$d\omega = \mathcal{A}(\partial_0 \vec{\omega} - \nabla \omega_0, \nabla \times \vec{\omega})$$

$$*: \Lambda^2 \to \Lambda^2$$

$$* \mathcal{F}(R, S) = \mathcal{F}(S, -R)$$

$$*d\, \mathcal{F}(R, S) = [\operatorname{div} S, -\nabla \times R + \partial_0 S]$$

$$*d* \mathcal{F}(R, S) = *d\, \mathcal{F}(S, -R)$$
$$= [-\operatorname{div} R, -\nabla \times S - \partial_0 R]$$

$$\begin{tikzcd}
 & \underset{\sim}{F} \arrow[dl] \arrow[dr] & \\
\hat{F} \arrow[d] & & F \arrow[d] \\
\hat{*}\hat{F} & & *F
\end{tikzcd}$$

$$L^t \hat{F} L = F$$

$$L^t \hat{*} \hat{F} L = *F$$

($*\underset{\sim}{F}$ defined by $*F$)

(well defined if $L^t \hat{*} \hat{F} L = *F$)

Much harder: $*dF$ transforms like a 1-form.

We define $\mathcal{M} : \Lambda^1 \to \Lambda^1$

$$\mathcal{M} = -\delta d$$

$$\boxed{\begin{aligned} &\text{Exercise: } -\delta d\,\omega = \Box\omega - d\delta\omega \\ &\qquad \text{where } \Box\omega = (\Box\omega_0, \dots, \Box\omega_3). \end{aligned}}$$

Fact: $\hat{\mathcal{M}}\,\hat{\omega}\,L = \mathcal{M}\omega$ ($\mathcal{M}\omega$ transforms as a 1-form)

$$\phi_\delta = \frac{1}{4\pi\varepsilon_0 |x|}$$

$$-\nabla \phi_\delta = E_\delta$$

$$\phi(x) = \int \phi_\delta(x-y)\rho(y)\,dy$$

$$-\nabla\phi = \int E_\delta(x-y)\rho(y)\,dy = E$$

$$-\Delta\phi = \int \operatorname{div} E_\delta \,\rho(y)\,dy$$

$$= \frac{1}{\varepsilon_0}\rho$$

$$d\omega = \mathfrak{A}(\partial_0 \vec{\omega} - \nabla\omega_0, \nabla\times\vec{\omega})$$

$$\omega = [\phi, 0]$$

$$d\omega = \mathfrak{A}(-\nabla\phi, 0) = \mathfrak{A}(E, 0)$$

in any frame

$$\boxed{d\omega = F}$$

$$d\omega = \mathfrak{A}(E, -cB)$$

$$-\delta\, d\omega = \Box\omega - d\,\overset{=0}{\delta\omega} \qquad \delta\omega = \partial_0\omega_0 - \mathrm{div}\,\vec{\omega} = 0$$

$$= (-\Delta\phi,\ 0)$$

$$= (-\mathrm{div}\,\nabla\phi,\ 0)$$

$$= \frac{1}{c\varepsilon_0}(c\rho,\ 0)$$

current density vector: $\begin{bmatrix} c\rho \\ j \end{bmatrix}$ $\quad c\rho \to C/m^3 \cdot \frac{m}{s}$, $\quad j \to \frac{C}{m^2}\frac{1}{s}$ charge flux

it's covector version

So in any frame, not just charges at rest,

$$\mathcal{M}\omega = \frac{1}{c\varepsilon_0}(c\rho,\ -j)$$

These are Maxwell's equations relating ω and charge density.

They hold in general, not just electrostatics.

$$d\omega = \mathcal{A}(E, -cB) \qquad \text{by def of } E, B.$$

$$* d\omega = \mathcal{A}(-cB, -E)$$

$$* d * d\omega = (-\operatorname{div} E,\ c\nabla \times B - \partial_0 S)$$

$$- * d * d\omega = (\operatorname{div} E,\ -c\nabla \times B + \partial_0 E)$$

$$* d\, \mathcal{A}(R, S) = [\operatorname{div} S,\ -\nabla \times R + \partial_0 S]$$

So ① $\operatorname{div} E = \dfrac{1}{\varepsilon_0} \rho$ (Gauss' Law)

$$-c\nabla \times B + \partial_0 E = -\frac{1}{c\varepsilon_0} j$$

② $\nabla \times B = \underbrace{\boxed{\dfrac{1}{c^2 \varepsilon_0}}}_{\mu_0} j - \underbrace{\dfrac{1}{c} \partial_0}_{\frac{1}{c^2}\partial_t} E$ (Ampere's equation with Maxwell's addition)

Exercise: $* d\, d\omega = 0$ (another face of $d^2 = 0$)

$$* d\, \mathcal{A}(E, -cB) = [-c \operatorname{div} B,\ -\nabla \times E + \partial_0 (-cB)]$$

③ $\operatorname{div} B = 0$ (no magnetic sources)

④ $\partial_t B + \nabla \times E = 0$ (Faraday's Law of Induction]

$$\delta\, \mathcal{F}(R,S) = *d\, \mathcal{F}(S,-R)$$
$$= [\operatorname{div}(-R),\ -\partial_0 R - \nabla \times S]$$
$$= -[\operatorname{div} R,\ \partial_0 R + \nabla \times S]$$

$$\delta\, [w_0, \vec{w}] = \partial_0 w_0 - \operatorname{div} \vec{w}$$

Exercise: $\delta^2 = 0$

But $\mathcal{M}\, \omega = -\delta d \omega = \underbrace{\frac{1}{c\varepsilon_0}[c\rho, -j]}_{\eta}$

$$\delta \eta = -\delta^2 d\omega = 0$$

$$J = \begin{bmatrix} c\rho \\ j \end{bmatrix}$$

$\operatorname{Div} J = 0 \longrightarrow$ conservation of charge.

$\int_\Omega \rho \leftarrow$ observed charge density

Gauge freedom.

Suppose $\mathcal{M}\,\omega = \frac{1}{c\varepsilon_0} j$

$$\tilde{\omega} = \omega + df$$

$$\begin{aligned}\mathcal{M}\,\tilde{\omega} &= -\delta d(\omega + df) \\ &= -\delta d\omega + -\delta d^2 f \\ &= \frac{1}{c\varepsilon_0} j + 0.\end{aligned}$$

Moreover, $d\tilde{\omega} = d\omega + d^2 f = d\omega$ so same EM field.

These are two faces of the same solution of Maxwell's equation and reflect a kind of coordinate change.

Now $\mathcal{M}\,\omega = \Box\omega - d\delta\omega$

If we can arrange $\delta\omega \equiv 0$ $\quad (\partial_0 \omega_0 - \operatorname{div} \vec{\omega} = 0)$

then Maxwell's equations reduce to wave equations

Is this even possible?

$$\tilde{\omega} = \omega + df$$

$$\delta \tilde{\omega} = \delta \omega + \delta d f$$

$$\Box f = -\delta \omega$$

So f needs to solve an inhomogeneous wave eq.

[We didn't discuss, but solution always exists, with arbitrary initial data]

I.e. if you can find one, you can find one with $\delta\tilde{\omega} = 0$.

This choice of gauge is called Lorentz gauge.

Start with arbitrary ω. $\partial_0 \omega$

Pick f a function of space alone,

$$-\Delta f = -\partial_0 \omega_0 + \operatorname{div} \vec{\omega}$$

So $\qquad \delta d f = -\delta \omega$ at $t = 0$.