We call $\vec{E} = \vec{E}_\delta q_0$ the electric field generated by e_0.

Let's drop the 'test''s.

$$\frac{d}{dt}\vec{p} = \vec{E} q$$

momentum of e_{test}

charge of e_{test}.

More generally, if e_1 and e_2 are charges at $\vec{x}_1, \vec{x}_2$ with charges q_1, q_2

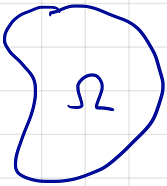
$$\vec{E} = E_\delta(\vec{x}-\vec{x}_1) q_1 + E_\delta(\vec{x}-\vec{x}_2) q_2.$$

And given a stationary charge density $\rho(x)$

$$\vec{E} = \int \vec{E}_\delta(\vec{x}-\vec{y}) \rho(\vec{y}) \, d\vec{y}$$

From HW:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$



$$\int_{\partial\Omega} E \cdot n = \frac{1}{\varepsilon_0} q := \frac{1}{\varepsilon_0} \int_{\Omega} \rho(\vec{y})\, d\vec{y}$$

total enclosed charge

Anyway,

$$\frac{d}{dt} \vec{p} = \vec{E} e$$

We interpret this as three components of $\frac{d}{dt} P$.

Can we deduce the full equation $\frac{d}{dt} P = ?$

and more naturally $\frac{d}{d\tau} P = ?$

↳ essentially 4 momentum.

Recall $P = m_0 V = m_0 \underbrace{\gamma(v)}_{\frac{dt}{d\tau}} \begin{bmatrix} c \\ \underbrace{\vec{v}}_{\frac{dx}{dt}} \end{bmatrix}$

$g(P, P) = m_0^2 c^2$ regardless of τ.

$$g\left(P, \frac{dP}{d\tau}\right) = 0$$

$$p^0 \frac{dp^0}{d\tau} - \underbrace{m_0 \gamma(v)}_{p^0/c} \vec{v} \cdot \frac{d\vec{p}}{d\tau} = 0$$

$$\frac{dp^0}{d\tau} = \frac{1}{c} \gamma(v) \vec{v} \cdot \frac{d\vec{p}}{dt}$$

$$= \frac{\gamma(v)}{c} v \cdot \vec{E} e$$

Also $\frac{d\vec{p}}{d\tau} = \gamma(v) \vec{E} e$

so $$\frac{d}{d\tau} P = \begin{bmatrix} \frac{\gamma(v)}{c} \; v \cdot E \\ \gamma(v) \vec{E} e \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} 0 & \vec{E}^T \\ \vec{E} & 0 \end{bmatrix} \gamma(v) \begin{bmatrix} c \\ \vec{v} \end{bmatrix} e$$

$$\frac{d}{d\tau} P = \frac{1}{c} \begin{bmatrix} 0 & \vec{E}^T \\ \vec{E} & 0 \end{bmatrix} V$$

For non-trivial reasons we'll factor

$$\begin{bmatrix} 0 & \vec{E}^T \\ \vec{E} & 0 \end{bmatrix} = G \underbrace{\begin{bmatrix} 0 & \vec{E}^T \\ -\vec{E} & 0 \end{bmatrix}}_{F}$$

F, antisymmetric.

electromagnetic field tensor.

$$c \frac{d}{d\tau} P = G F V e$$

Now move to a frame where charges are moving.

$$\hat{x} = Lx$$

$$\hat{P} = LP$$

$$\hat{V} = LV$$

I claim if $L^T \hat{F} L = F$ then

$$c\frac{d}{d\tau}\hat{P} = G\hat{F}\hat{V}e \qquad \text{as well.}$$

Indeed $\qquad L^t G L = G$

$$GFVe = GL^t\hat{F}LVe \qquad GL^t = L^{-1}G$$

$$= L^{-1}GL^t\hat{F}LVe$$

$$c\frac{d}{d\tau}\hat{P} = c\frac{d}{d\tau}LP = L(GFVe)$$

$$= L^{-1}LG\hat{F}\hat{V}e$$

$$= G\hat{F}\hat{V}e.$$

We call $\hat{F}$ the E-M field in the boosted frame.

Coordinat free vesion

$$\underset{\sim}{F}(\underset{\sim}{X}, \underset{\sim}{Y}) = X^t F Y$$

in any other coord system,

$$\begin{aligned} \hat{X}^t \hat{F} \hat{Y} &= (LX)^t \hat{F} L Y \\ &= X^t L^t \hat{F} L Y \\ &= X^T F Y \end{aligned}$$

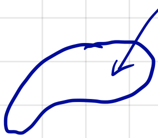
Also, $L^t \hat{F} L = F \to (L^t)^{-1} F L^{-1} = \hat{F}$

$$L^t \hat{F}^t L = -F$$

$$\hat{F}^t = -(L^t)^{-1} F L^{-1} = -\hat{F},$$ so

always anti symmetric

$$\underset{\sim}{F}(\underset{\sim}{X}, \underset{\sim}{Y}) = -\underset{\sim}{F}(\underset{\sim}{Y}, \underset{\sim}{X}).$$



in fact it's designed to be integrated over 2-d surfaces in spacetime, but I'm getting ahead of myself.

$$F = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ -E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}$$ in any coordinate system.

This is just a convention on the naming of the entries and agrees with the stationary case.

$$\vec{E} = (E_1, E_2, E_3)$$
$$\vec{B} = (B_1, B_2, B_3)$$
are called electric, magnetic field.

They may look like vectors (and they transform as 3-vectors under rotation) to you, but they do not transform like anything useful under boosts.

Only F obeys the nice transformation law.

$$\begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -B_3 v_2 + B_2 v_3 \\ B_3 v_1 - B_1 v_2 \\ -B_2 v_1 + B_1 v_2 \end{bmatrix}$$

$$\begin{matrix} B_1 & B_2 & B_3 \\ v_1 & v_2 & v_3 \end{matrix} \qquad = \quad \vec{B} \times \vec{v} = -\vec{v} \times B$$

So $\quad F \begin{bmatrix} c \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{E} \cdot \vec{v} \\ -c\vec{E} - c\vec{v} \times \vec{B} \end{bmatrix}$

$$c \frac{dP}{d\tau} = \gamma(v) G F \begin{bmatrix} c \\ \vec{v} \end{bmatrix} e = \gamma(v) \begin{bmatrix} \vec{E} \cdot \vec{v} \\ +c\vec{E} + c\,\vec{v} \times \vec{B} \end{bmatrix} e$$

If $|v| \ll c$, $\frac{d}{d\tau} \approx \frac{d}{dt}$, $\quad P \approx \begin{bmatrix} \text{energy} \\ m\vec{v} \end{bmatrix}$

$\frac{d}{dt} m\vec{v} = \vec{E}e + \vec{v} \times Be \quad \Big]$ Lorentz force law.

$\frac{d}{dt}\,\text{energy} = \frac{1}{c} \vec{E} \cdot \vec{v} e$ does not involve mag field.

$\vec{v} \times B \perp \vec{v}$

To this point we have not used the relation

$$\vec{E} = \int_{\mathbb{R}^3} \vec{E}_\delta(x-y)\,\rho(y)\,dy \qquad \vec{E}_\delta = \frac{1}{4\pi\varepsilon_0}\frac{1}{|x|^2}\frac{x}{|x|}$$

Next goal: "derive" Maxwell's equations.

Notation: V: spacetime.

Λ^0 functions on spacetime

Λ^1 covector fields on spacetime

$$d: \Lambda^0 \longrightarrow \Lambda^1$$

In coords, $df = [\partial_0 f, \dots, \partial_3 f]$

It has a companion

$$\delta: \Lambda^1 \rightarrow \Lambda^0$$

$$\delta\omega = \partial_0\omega_0 - \partial_1\omega_1 - \partial_2\omega_2 - \partial_3\omega_3$$

$$\omega \longrightarrow V \longrightarrow \mathrm{Div}\cdot V = \delta\omega$$

Exercise: $\boxed{\hat{\delta}\hat{\omega} = \delta_\omega}$

$$\delta d f = \Box f$$

Λ^1: at each $x \in U$, is a map $\omega : V \to \mathbb{R}$.

Λ^2: at each $x \in U$, is a map $F: V \times V \to \mathbb{R}$,

bilinear, $F(v,w) = -F(w,v)$.

(e.g. the E-M field). We can rep. by an antisymmetric matrix.

Λ^3, Λ^4 as well.

And each of these is meant to be integrated

(Λ^1 over lines, Λ^2 over 2-surfaces, etc).

$\int_a^b \eta(\alpha'(s))ds$ is independent of param, but depends on direction.

Big picture

$$\begin{tikzcd}
\Lambda^0 \arrow[r, shift left, "d"] & \Lambda^1 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Lambda^2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Lambda^3 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Lambda^4 \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

we've seen this

we'll visit this for Maxwell

Moreover $d^2 = 0$

Hodge $*$ $\quad * : \Lambda^i \to \Lambda^{4-i}$

$$\begin{aligned}
* &: \Lambda^0 \to \Lambda^4 \\
&: \Lambda^1 \to \Lambda^3 \\
&: \Lambda^2 \to \Lambda^2 \\
&: \Lambda^3 \to \Lambda^1 \\
&: \Lambda^4 \to \Lambda^0
\end{aligned}$$

$\delta = * d *$ $\quad$ e.g. $\Lambda^2 \to \Lambda^2 \to \Lambda^3 \to \Lambda^1$

(so $\delta^2 = 0$) $\quad$ $\Lambda^1 \to \Lambda^3 \to \Lambda^4 \to \Lambda^0$

But I'm going to try to avoid discussing Λ^3, Λ^4
($\Lambda^0, \Lambda^1, \Lambda^2$ can be rep in terms of matrices, vectors)

$$\Lambda^0 \underset{\delta}{\overset{d}{\rightleftarrows}} \Lambda^1 \qquad \delta d = \Box$$

$$\Lambda^1 \underset{\delta}{\overset{d}{\rightleftarrows}} \Lambda^2 \qquad -\delta d = \mathcal{M}$$ $\rightarrow$ maxwell operator.

So who is d, ∂?

$$\omega = [\omega_0, \omega_1, \omega_2, \omega_3]$$

$$(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i$$

Exercise $L^t \hat{d}\hat{\omega} L = d\omega$ ($d\omega$ transforms like a 2-form)

Exercise $d^2 : \Lambda^0 \to \Lambda^2 = 0$

This is a deep generalization of $\nabla \times (\nabla f) = 0$

$$\operatorname{div}(\nabla \times V) = 0$$

To describe S I need some notation.

An antisymmetric 4x4 matrix has 6 independent entries.

Given
$$R = [R_1, R_2, R_3]$$
$$S = [S_1, S_2, S_3]$$

$$\mathcal{A}(R, S) = \begin{bmatrix} 0 & R_1 & R_2 & R_3 \\ -R_1 & 0 & S_3 & -S_2 \\ -R_2 & -S_3 & 0 & S_1 \\ -R_3 & S_2 & -S_1 & 0 \end{bmatrix}$$

Gives us a way to talk about them.

e.g. $F = \mathcal{A}(E, -cB)$

Exercise: if $\omega = [\omega_0, \vec{\omega}]$

$$d\omega = \mathcal{A}(\partial_0 \vec{\omega} - \nabla \omega_0, \nabla \times \vec{\omega})$$

$$*: \Lambda^2 \to \Lambda^2$$

$$* \mathcal{F}(R, S) = \mathcal{F}(S, -R)$$

$$\begin{tikzcd}
 & \underset{\sim}{F} \arrow[dl] \arrow[dr] & \\
\hat{F} \arrow[d] & & F \arrow[d] \\
\hat{*}\hat{F} & & *F
\end{tikzcd}$$

$$L^t \hat{F} L = F$$

$$L^t \hat{*} \hat{F} L = *F$$

($*\underset{\sim}{F}$ defined by $*F$)

(well defined if $L^t \hat{*} \hat{F} L = *F$)

$$*d\, \mathcal{F}(R, S) = [\operatorname{div} S, -\nabla \times R + \partial_0 S]$$

$$*d* \mathcal{F}(R, S) = *d\, \mathcal{F}(S, -R)$$

$$= [-\operatorname{div} R, -\nabla \times S - \partial_0 R]$$

We define $\mathcal{M} : \Lambda^1 \to \Lambda^1$

$$\mathcal{M} = -\delta d$$

Exercise: $-\delta d\,\omega = \Box\omega - d\delta\omega$

where $\Box\omega = (\Box\omega_0, \ldots, \Box\omega_3)$.

Fact: $\hat{\mathcal{M}}\,\hat{\omega}\,L = \mathcal{M}\omega$ ($\mathcal{M}\omega$ transforms as a 1-form)

$$\phi_\delta = \frac{1}{4\pi\varepsilon_0 |x|}$$

$$-\nabla \phi_\delta = E_\delta$$

$$\phi(x) = \int \phi_\delta(x-y)\,\rho(y)\,dy$$

$$-\nabla \phi = \int E_\delta(x-y)\,\rho(y)\,dy = E$$

$$-\Delta \phi = \int \operatorname{div} E_\delta \,\rho(y)\,dy$$

$$= \frac{1}{\varepsilon_0}\rho$$

$$\omega = [\phi, 0]$$

$$d\omega = \mathfrak{A}(-\nabla\phi, 0) = \mathfrak{A}(E, 0)$$

in any frame $\boxed{d\omega = F}$

$$d\omega = \mathfrak{A}(E, -cB)$$

$$-\delta d\omega = \Box\omega - d\underbrace{\delta\omega}_{=0}$$

$$= (-\Delta\phi, 0)$$

$$= (-\operatorname{div}\nabla\phi, 0)$$

$$= \frac{1}{c\varepsilon_0}(c\rho, 0)$$

current density vector:

$$\begin{bmatrix} c\rho \\ j \end{bmatrix}$$

$C/m^3 \cdot \frac{m}{s}$

$\frac{C}{m^2}\frac{1}{s}$

charge flux

it's covector version

So in any frame, not just at rest charges,

$$\mathcal{M}\omega = \frac{1}{c\varepsilon_0}(c\rho, -j)$$