But also, any linear combination of these is again a solution of the wave equation. It's a sum of plane waves being transported in various directions at speed c.

"Energy" density of a solution

$$\frac{1}{2}\left(\frac{1}{c^2}u_t^2 + |\nabla u|^2\right)$$

$$\frac{d}{dt}\int \frac{1}{2}\left(\frac{1}{c^2}u_t^2 + |\nabla u|^2\right) dx$$

$$= \int \frac{1}{c^2}u_t u_{tt} + \nabla u \cdot \nabla u_t \, dx$$

$$= \int \frac{1}{c^2} u_t u_{tt} - u_t \Delta u \, dx$$

$$= \int u_t \left[u_{tt} - \Delta u\right] dx = 0.$$

In fact, the solution has an associated energy-momentum.

$$P = \begin{bmatrix} \frac{1}{2}\left(\frac{|u_t|^2}{c^2} + |\nabla u|^2\right) \\ -\frac{1}{c} u_t \nabla u \end{bmatrix}$$

$$\operatorname{Div} P = \frac{1}{c^3} u_t u_{tt} + \frac{1}{c} \nabla u_t \cdot \nabla u - \frac{1}{c} \nabla u_t \cdot \nabla u - \frac{1}{c} u_t \Delta u$$

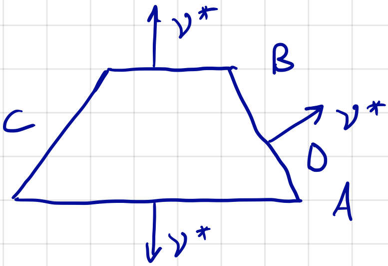
$$= \frac{1}{c} u_t \left[\frac{1}{c^2} u_{tt} - \Delta u \right] = 0.$$

This is a locally conserved quantity.

It is causal, future pointing by Cauchy-Schwartz.

A 2-d computation

$$P = \left[\frac{1}{2}\left(\frac{1}{c^2}\partial_t u\right)^2 + (\partial_x u)^2 \right]$$



(Euclidean!)

$$0 = -\int_A \frac{1}{2}\left[\frac{1}{c^2}(\partial_t u)^2 + (\partial_x u)^2\right] + \int_B \frac{1}{2}\left[\text{——}\right]$$

$$= \int_C \nu^* \cdot P + \int_D \nu^* \cdot P$$

On D $\quad \nu^* = \frac{1}{\sqrt{2}}[1, 1]$

$$\nu^* \cdot P = g(\nu, P) \qquad \nu = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ν

On C $\quad \nu^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\nu = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

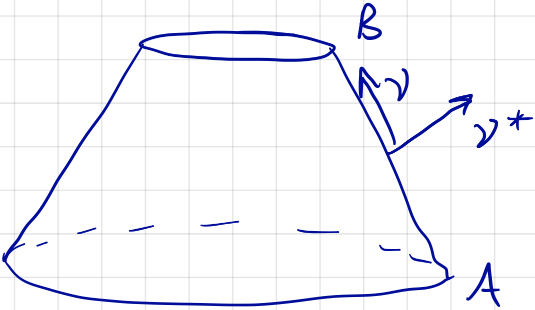
$g(P, \nu) \geq 0$ as P is time like, f.p
ν is f.p, null

$$\int_B \frac{1}{2}\left[\text{——} \right] = \int_A \frac{1}{2}\left[\text{——} \right] - \int_{C \cup D} g(P, \nu)\, dA$$

$$\leq \int_A \frac{1}{2}\left[\text{——} \right]$$

(with equality iff
P is parallel to ν)

Higher dimens:



$$\int_B \frac{1}{2}[\;\text{—}\;] = \int_A \frac{1}{2}[\;\text{—}\;] - \underbrace{\int_C g(\nu, P)}_{\leq 0}$$

If $\partial_t u$, $\nabla u = 0$ on A then also on B

Exercise: u is constant on whole frustum.

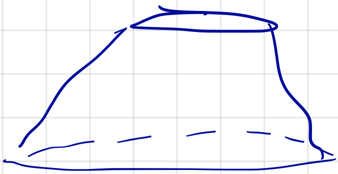
Cor: There exists at most one solution of

$$\Box u = 0$$
$$u\big|_{t=0} = \phi$$
$$\frac{1}{c} u_t \big|_{t=0} = \psi$$

Pf: If u_1, u_2 are solutions then $w = u_2 - u_1$

solves
$$\Box w = 0$$
$$w\big|_{t=0} = 0$$
$$\frac{1}{c} w_t \big|_{t=0} = 0$$

Then on any disc



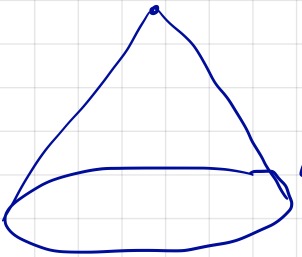
$w_t^2 + |Dw|^2 = 0$

so w is constant.

So w is independent of space $w = w(t)$. But $w_t \equiv 0$ and w is constant. But $w\big|_{t=0} = 0$ so $w = 0$.

In fact, under mild hypotheses on ϕ, ψ, there exists a solution:

$$u(x,t) = \frac{1}{c\,4\pi t} \int_{\partial B(x,ct)} \psi(y)\,dy + \frac{1}{c}\partial_t \left[\frac{1}{4\pi t c} \int_{\partial B(x,ct)} \phi(y)\,dy \right]$$



depends on ψ, ϕ, $\nabla\phi$ here right at the boundary.

Maxwell's Equations

Particles in nature posess an innate property called charge, in the same sense that particles also posess mass.

It's measured in Coulombs, which is the charge of 6.2×10^{8} electrons.

Classical observation: two particles e_0, e_{test} with charge q_0 q_{test}

• e_{test}

↳ at $\vec{x}$

• e_0

↳ stationary at the origin

The force exerted by e_0 on e_{test} is

$$\vec{F} = \vec{E}_{\delta}\, q_0 q_{test}$$

Where

$$\vec{E}_{\delta} = \frac{1}{4\pi\varepsilon_0 |\vec{x}|^2} \frac{\vec{x}}{|\vec{x}|}$$

This is true ind of velocity of e_{test}, and is called the Coulomb force law.

We call $\vec{E} = \vec{E}_\delta q_0$ the electric field generated by e_0.

Let's drop the 'test''s.

$$\frac{d}{dt}\vec{p} = \vec{E} q$$

momentum of e_{test}

charge of e_{test}.

More generally, if e_1 and e_2 are charges at $\vec{x}_1, \vec{x}_2$ with charges q_1, q_2

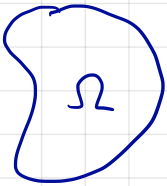
$$\vec{E} = E_\delta(\vec{x}-\vec{x}_1)q_1 + E_\delta(\vec{x}-\vec{x}_2)q_2.$$

And given a <u>stationary</u> charge density $\rho(x)$

$$\vec{E} = \int \vec{E}_\delta(\vec{x}-\vec{y})\rho(\vec{y})\,d\vec{y}$$

From HW:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$



$$\int_{\partial\Omega} E \cdot n = \frac{1}{\varepsilon_0} \underbrace{q}_{\text{total enclosed charge}} := \frac{1}{\varepsilon_0} \int_{\Omega} \rho(\vec{y})\, d\vec{y}$$

Anyway,

$$\frac{d}{dt} \vec{p} = \vec{E} e$$

We interpret this as three components of $\frac{d}{dt} P$.

Can we deduce the full equation $\frac{d}{dt} P = ?$

and more naturally $\frac{d}{d\tau} P = ?$

↳ essentially 4 momentum.

Recall $P = m_0 V = m_0 \underbrace{\gamma(v)}_{\frac{dt}{d\tau}} \begin{bmatrix} c \\ \underbrace{\vec{v}}_{\frac{dx}{dt}} \end{bmatrix}$

$g(P, P) = m_0^2 c^2$ regardless of τ.

$$g\left(P, \frac{dP}{d\tau}\right) = 0$$

$$P^0 \frac{dP^0}{d\tau} - \underbrace{m_0 \gamma(v)}_{P^0/c} \vec{v} \cdot \frac{d\vec{p}}{d\tau} = 0$$

$$\frac{dP^0}{d\tau} = \frac{1}{c} \gamma(v) \vec{v} \cdot \frac{d\vec{p}}{dt}$$

$$= \frac{\gamma(v)}{c} v \cdot \vec{E} e$$

Also $\frac{d\vec{p}}{d\tau} = \gamma(v) \vec{E} e$

so $$\frac{d}{d\tau} P = \begin{bmatrix} \frac{\gamma(v)}{c} \, v \cdot E \\ \gamma(v) \, \vec{E} \, e \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} 0 & \vec{E}^T \\ \vec{E} & 0 \end{bmatrix} \gamma(v) \begin{bmatrix} c \\ \vec{v} \end{bmatrix} e$$

$$\frac{d}{d\tau} P = \frac{1}{c} \begin{bmatrix} 0 & \vec{E}^T \\ \vec{E} & 0 \end{bmatrix} V$$

For non-trivial reasons we'll factor

$$\begin{bmatrix} 0 & \vec{E}^T \\ \vec{E} & 0 \end{bmatrix} = G \underbrace{\begin{bmatrix} 0 & \vec{E}^T \\ -\vec{E} & 0 \end{bmatrix}}_{F}$$

F, antisymmetric.

electromagnetic field tensor.

$$c \frac{d}{d\tau} P = G F V e$$

Now move to a frame where charges are moving.

$$\hat{x} = Lx$$

$$\hat{p} = Lp$$

$$\hat{V} = LV$$

I claim if $L^T \hat{F} L = F$ then

$$c \frac{d}{d\tau} \hat{p} = G \hat{F} \hat{V} e$$ as well.

Indeed $\qquad L^t G L = G$

$$G F V e = G L^t \hat{F} L V e \qquad G L^t = L^{-1} G$$

$$= L^{-1} G L^t \hat{F} L V e$$

$$c \frac{d}{d\tau} \hat{p} = c \frac{d}{d\tau} L p = L (G F V e)$$

$$= L^{-1} L G \hat{F} \hat{V} e$$

$$= G \hat{F} \hat{V} e.$$

We call $\hat{F}$ the E-M field in the boosted frame.

Coordinat free vesion

$$\underset{\sim}{F}(\underset{\sim}{X}, \underset{\sim}{Y}) = X^t F Y$$

in any other coord system,

$$\hat{X}^t \hat{F} \hat{Y} = (LX)^t \hat{F} L Y$$
$$= X^t L^t \hat{F} L Y$$
$$= X^T F Y$$

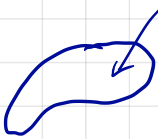
Also, $L^t \hat{F} L = F \longrightarrow (L^t)^{-1} F L^{-1} = \hat{F}$

$$L^t \hat{F}^t L = -F$$

$$\hat{F}^t = -(L^t)^{-1} F L^{-1} = -\hat{F},$$ so

always anti symmetric

$$\underset{\sim}{F}(\underset{\sim}{X}, \underset{\sim}{Y}) = -\underset{\sim}{F}(\underset{\sim}{Y}, \underset{\sim}{X}).$$



in fact it's designed to be integrated over 2-d surfaces in spacetime, but I'm getting ahead of myself.

$$F = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ -E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}$$ in any coordinate system.

This is just a convention on the naming of the entries and agrees with the stationary case.

$$\vec{E} = (E_1, E_2, E_3)$$
$$\vec{B} = (B_1, B_2, B_3)$$
are called electric, magnetic field.

They may look like vectors to you, but they do not transform like vectors. F has the nice transformation law.

$$\begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -B_3 v_2 + B_2 v_3 \\ B_3 v_1 - B_1 v_2 \\ -B_2 v_1 + B_1 v_2 \end{bmatrix}$$

$$\begin{matrix} B_1 & B_2 & B_3 \\ v_1 & v_2 & v_3 \end{matrix} \qquad = \vec{B} \times \vec{v} = -\vec{v} \times B$$

So $$F \begin{bmatrix} c \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{E} \cdot \vec{v} \\ -c\vec{E} - c\vec{v} \times \vec{B} \end{bmatrix}$$

$$c \frac{dP}{d\tau} = \gamma(v) G F \begin{bmatrix} c \\ \vec{v} \end{bmatrix} e = \gamma(v) \begin{bmatrix} \vec{E} \cdot \vec{v} \\ +c\vec{E} + c \, \vec{v} \times \vec{B} \end{bmatrix} e$$

If $|v| \ll c$, $\frac{d}{d\tau} \approx \frac{d}{dt}$, $P \approx \begin{bmatrix} \text{energy} \\ m\vec{v} \end{bmatrix}$

$$\left. \frac{d}{dt} m\vec{v} = \vec{E}e + \vec{v} \times Be \right] \quad \text{Lorentz force law.}$$

$$\frac{d}{dt} \text{energy} = \frac{1}{c} \vec{E} \cdot \vec{v} e$$ does not involve mag field.

$\vec{v} \times B \perp \vec{v}$